J11Q.1 Solution

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Part a) asks for the probability of measuring $\sigma_z = -\hbar/2$ when it starts out as $\sigma_z = +\hbar/2$.

$$
\begin{aligned}
|\psi(0)> &= |\frac{1}{2}, +\frac{1}{2}> \\
\Rightarrow |\psi(t)> &= \hat{U}(t)|\psi(0)> = e^{-ib\sigma_x t}|\frac{1}{2}, +\frac{1}{2}> \\
|\frac{1}{2}, +\frac{1}{2}>_z &= \frac{1}{\sqrt{2}}|\frac{1}{2}, +\frac{1}{2}>_x + \frac{1}{\sqrt{2}}|\frac{1}{2}, -\frac{1}{2}>_x \\
|\psi(t)> &= \frac{e^{-ibt}}{\sqrt{2}}|\frac{1}{2}, +\frac{1}{2}>_x + \frac{e^{ibt}}{\sqrt{2}}|\frac{1}{2}, -\frac{1}{2}>_x \\
|\frac{1}{2}, +\frac{1}{2}>_x &= \frac{1}{\sqrt{2}}|\frac{1}{2}, +\frac{1}{2}>_z + \frac{1}{\sqrt{2}}|\frac{1}{2}, -\frac{1}{2}>_z \\
|\frac{1}{2}, -\frac{1}{2}>_x &= \frac{1}{\sqrt{2}}|\frac{1}{2}, +\frac{1}{2}>_z - \frac{1}{\sqrt{2}}|\frac{1}{2}, -\frac{1}{2}>_z \\
<\frac{1}{2}, -\frac{1}{2}|\psi(t)> &= \frac{e^{-ibt}}{2} - \frac{e^{ibt}}{2} \\
\Rightarrow |<\frac{1}{2}, -\frac{1}{2}|\psi(t)>|^2 &= \sin^2(bt)
\end{aligned}
$$

Part b) is asking us to find the probability of finding $\sigma_z = -\hbar/2$ after a second measurement a time τ later. From part a) we know that

$$
|<\frac{1}{2}, +\frac{1}{2}|\psi(t)>|^2 = \cos^2(bt)
$$

since the probabilities must add to unity. If the particle is measured as $\sigma_z = -\hbar/2$ at time t_1 then we have to use $|\psi(0)> = |\frac{1}{2}, -\frac{1}{2}>$ to find the probabilities at some time t later.

$$
\begin{aligned}
|\psi(0)>=&|\frac{1}{2},-\frac{1}{2}> \\
\Rightarrow |\psi(t)>=&\hat{U}(t)|\psi(0)>= e^{-ib\sigma_x t}|\frac{1}{2},-\frac{1}{2}> \\
|\frac{1}{2},-\frac{1}{2}>_z=&\frac{1}{\sqrt{2}}|\frac{1}{2},+\frac{1}{2}>_x-\frac{1}{\sqrt{2}}|\frac{1}{2},-\frac{1}{2}>_x \\
|\psi(t)>=&\frac{e^{-ibt}}{\sqrt{2}}|\frac{1}{2},+\frac{1}{2}>_x-\frac{e^{ibt}}{\sqrt{2}}|\frac{1}{2},-\frac{1}{2}>_x \\
|\frac{1}{2},+\frac{1}{2}>_x=&\frac{1}{\sqrt{2}}|\frac{1}{2},+\frac{1}{2}>_z+\frac{1}{\sqrt{2}}|\frac{1}{2},-\frac{1}{2}>_z \\
|\frac{1}{2},-\frac{1}{2}>_x=&\frac{1}{\sqrt{2}}|\frac{1}{2},+\frac{1}{2}>_z-\frac{1}{\sqrt{2}}|\frac{1}{2},-\frac{1}{2}>_z \\
<\frac{1}{2},-\frac{1}{2}|\psi(t)>=&\frac{e^{-ibt}}{2}+\frac{e^{ibt}}{2} \\
\Rightarrow |<\frac{1}{2},-\frac{1}{2}|\psi(t)>|^2=&\cos^2(bt)
\end{aligned}
$$

The total probability of measuring $\sigma_z = -\hbar/2$ at time $t_2 = t_1+\tau$ is the probability of measuring $\sigma_z = +\hbar/2$ then $\sigma_z = -\hbar/2$ plus the probability of measuring $\sigma_z = -\hbar/2$ then $\sigma_z = -\hbar/2$. Thus the total probability is:

$$\sin^2(bt_1)\cos^2(b\tau)+\cos^2(bt_1)\sin^2(b\tau)$$

Part c) asks for the probability of measuring the $\sigma_z = -\hbar/2$ state after n measurements separated by τ in time. From the work done above the $n = 3$ case is clearly

$$
\begin{aligned}
&\sin^2(b\tau)\cos^2(b\tau)\cos^2(b\tau)+\cos^2(b\tau)\sin^2(b\tau)\cos^2(b\tau) \\
&+\sin^2(b\tau)\sin^2(b\tau)\sin^2(b\tau)+\cos^2(b\tau)\cos^2(b\tau)\sin^2(b\tau)
\end{aligned}
$$

Since $b\tau << 1$ we can expand this so $\sin^2(b\tau) \approx (b\tau)^2$ and $\cos^2(b\tau) \approx 1$. Thus for n=3

$$P_3 = 3(b\tau)^2 + O((b\tau)^4)$$

Let's prove this by induction, I propose the solution is the form

$$P_n = n(b\tau)^2$$

Then, my inductive hypothesis would be

$$P_{n-1} = (n-1)(b\tau)^2$$

Thus the probability of getting $\sigma_z = +\hbar/2$ must be

$$P'_{n-1} = 1-(n-1)(b\tau)^2$$

If we got $\sigma_z = +\hbar/2$ in the $n-1$ case then we need to multiply by $\sin^2(b\tau) \approx (b\tau)^2$ to get the chance of $\sigma_z = -\hbar/2$ but if we got $\sigma_z = -\hbar/2$ at $n-1$ then we need to multiply by $\cos^2(b\tau) \approx 1$. Thus the probability of getting $\sigma_z = -\hbar/2$ is approximately

$$P_n = (b\tau)^2 P'_{n-1} + P_{n-1}$$

Thus the probability to get $\sigma_z = -\hbar/2$ is

$$P_n = n(b\tau)^2$$

By induction. Putting 0.1 in for P_n and T/τ for n we get that τ must be

$$\begin{aligned} 0.1 &> (T/\tau)(b\tau)^2 \\ \tau &< \frac{0.1}{T \cdot b^2} \end{aligned}$$