J11E.1 Solution

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a) luckily for me I remembered several things that were very helpful for this problem. Most importantly

$$\langle \vec{S} \rangle = \frac{1}{2}\vec{E}x\vec{H}^*$$

Then, knowing your trusty wave solutions to Maxwell's equations in a coaxial cable(since k isn't specified in the problem let $k = \omega/c$) you can solve for the time averaged Poynting vector

$$\vec{E} = \frac{E_0}{s}e^{i(kz-\omega t)}\hat{s}$$
$$\vec{B} = \frac{E_0}{sc}e^{i(kz-\omega t)}\hat{\phi}$$
$$\langle \vec{S} \rangle = \frac{E_0^2}{2s^2c\mu_0}\hat{z}$$

then relate that to the time averaged power radiated via

$$\langle P \rangle = \int \langle \vec{S} \rangle \cdot d\vec{A}$$
$$\langle P \rangle = \int_0^{2\pi}\int_a^b \frac{E_0^2}{2s^2c\mu_0} s ds d\phi$$
$$\langle P \rangle = \frac{E_0^2\pi}{2c\mu_0}\log(b/a)$$

b) A good thing to memorize from Griffith's is the relation between the incident, reflected, and transmitted wave amplitudes

$$E_R = \frac{1-\beta}{1+\beta}E_I$$
$$E_T = \frac{2}{1+\beta}E_I$$

Where $\beta = \frac{\mu_1 v_1}{\mu_2 v_2}$. so for situations like this where $\mu_1 \approx \mu_2$ (which is most materials by the way) $\beta \approx \sqrt{\epsilon_r}$ where $\epsilon_r = \frac{\epsilon}{\epsilon_0}$

$$
\begin{aligned}
\vec{E_R} =& \frac{E_0}{s} \frac{1-\sqrt{\epsilon_r}}{1+\sqrt{\epsilon_r}} e^{i(kz-\omega t)} \hat{s} \\
\vec{B_R} =& -\frac{E_0}{sc} \frac{1-\sqrt{\epsilon_r}}{1+\sqrt{\epsilon_r}} e^{i(kz-\omega t)} \hat{\phi}
\end{aligned}
$$

Where the negative on $\vec{B_R}$ is from the flipping at the boundary. This is necessary since otherwise the poynting vector would still be propagating in the $+\hat{z}$ direction

$$
\begin{aligned}
\vec{E_T} =& \frac{E_0}{s} \frac{2}{1+\sqrt{\epsilon_r}} e^{i(kz-\omega t)} \hat{s} \\
\vec{B_T} =& \frac{E_0}{sc} \sqrt{\epsilon_r} \frac{2}{1+\sqrt{\epsilon_r}} e^{i(kz-\omega t)} \hat{\phi}
\end{aligned}
$$

Note that the k in the transmitted wave functions are not the same as in the reflected since in the transmitted waves $k = \omega\sqrt{\epsilon_r}/c$

c) To find the force per unit area all we need is the net radiation pressure from all these EM-waves. This is given by

$$
\begin{aligned}
P_{rad} =& \frac{|\langle \vec{S} \rangle|}{c} \\
P_{rad} =& \frac{|\langle \vec{S_I} \rangle|}{c} - \frac{|\langle \vec{S_R} \rangle|}{c} - \frac{|\langle \vec{S_T} \rangle|}{c} \\
P_{rad} =& \frac{E_0^2}{2s^2c^2\mu_0} - \frac{E_0^2}{2s^2c^2\mu_0} \left(\frac{1-\sqrt{\epsilon_r}}{1+\sqrt{\epsilon_r}} \right)^2 - \frac{E_0^2\sqrt{\epsilon_r}}{2s^2c^2\mu_0} \left(\frac{2}{1+\sqrt{\epsilon_r}} \right)^2
\end{aligned}
$$

Some algebra yields the answer

$$
\begin{aligned}
P_{rad} =& \frac{E_0}{s^2c^2\mu_0} \left(\frac{1-\sqrt{\epsilon_r}}{1+\sqrt{\epsilon_r}} \right)^2 \\
\Rightarrow \vec{F}_{total} =& \int_0^{2\pi} \int_a^b \frac{E_0}{s^2c^2\mu_0} \left(\frac{1-\sqrt{\epsilon_r}}{1+\sqrt{\epsilon_r}} \right)^2 s ds d\phi \hat{z} \\
\vec{F}_{total} =& \frac{E_0}{c^2\mu_0} \left(\frac{1-\sqrt{\epsilon_r}}{1+\sqrt{\epsilon_r}} \right)^2 \log(b/a) \hat{z} \\
\vec{F}_{average} =& \frac{E_0}{\pi(b^2-a^2)c^2\mu_0} \left(\frac{1-\sqrt{\epsilon_r}}{1+\sqrt{\epsilon_r}} \right)^2 \log(b/a) \hat{z}
\end{aligned}
$$