Problem J10Q.1

(a) i. The isotropic oscillator potential is separable along the three coordinate axes, so the eigenstates are parameterized by the occupation numbers n_x, n_y, n_z along each axis. The corresponding eigen-energies are

$$E_{n_x,n_y,n_z} = \hbar\omega\left(n_x + n_y + n_z + \frac{3}{2}\right).$$

By stars and bars, the n^{th} energy level (indexed from $n = 0$) of the oscillator has energy

$$E_n = \hbar\omega\left(\frac{3}{2} + n\right),$$

and has degeneracy

$$\binom{n+2}{2} = \frac{(n+2)(n+1)}{2}.$$

The table below lists the energies and degeneracies of the first three levels.

n	E_n	Degeneracy
0	$\frac{3}{2}\hbar\omega$	1
1	$\frac{5}{2}\hbar\omega$	3
2	$\frac{7}{2}\hbar\omega$	6

ii. The $n = 0$ level must transform under the spin-0 representation, by dimension count.

The $n = 1$ level has odd spatial parity, so it must transform under an odd-integer spin representation. By dimension count, the only possiblity is one irreducible spin-1 multiplet.

The $n = 2$ level has even spatial parity, and so all its irreducible representations must have even-integer spin. The only possibilities are six spin-0 representations, or one spin-2 and one spin-0 representation.

Since it is possible to rotate the $|2,0,0\rangle$ state into a $|0,2,0\rangle$ state, there must exist a spin representation of dimension ≥ 2, ruling out the first option. Thus the $n = 2$ level must contain one spin-2 multiplet and one spin-0 singlet state (i.e. $\mathbf{2} \oplus \mathbf{0}$).

(b) Since the problem is isotropic, we may assume that $\mathbf{b} = (b, 0, 0)$ lies along the x-axis, so that

$$H' = \lambda b^3 \left(\frac{\hbar}{2m\omega}\right)^{3/2} \left(a + a^\dagger\right)^3,$$

where $a, a^\dagger$ are the ladder operators for the x oscillator. Let $g := |0,0,0\rangle$ denote the ground state.

Since every term in H' contains an odd number of ladder operators, the first-order energy shift $\langle g|H'|g\rangle = 0$ vanishes.

To evaluate the second-order contribution, we compute the matrix elements

$$\langle 1,0,0|(a + a^\dagger)^3|g\rangle = \langle 1,0,0|(a^\dagger a a^\dagger + a a^\dagger a^\dagger)|g\rangle = 3$$

and

$$\langle 3,0,0|(a + a^\dagger)^3|g\rangle = \langle 3,0,0|(a^\dagger)^3|g\rangle = \sqrt{6}.$$

Then the second-order shift is

$$\begin{aligned} E_0^{(2)} &= -\frac{|\langle 1,0,0|H'|g\rangle|^2}{\hbar\omega} - \frac{|\langle 3,0,0|H'|g\rangle|^2}{3\hbar\omega} \\ &= -\frac{11}{\hbar\omega}\lambda b^3 \left(\frac{\hbar}{2m\omega}\right)^3 \\ &= \boxed{-\frac{11\hbar^2}{8m^3\omega^4}\lambda b^3.} \end{aligned}$$

(c) Again by isotropy, we may assume that $\mathbf{E}_0 = (E_0, 0, 0)$ is aligned along the x-axis. Then the perturbation gives a contribution

$$H' = qE_0 e^{-t/\tau}\sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)\,\Theta(t)$$

to the Hamiltonian.

In first-order perturbation theory, the probability of a transition to the state $|n, 0, 0\rangle$ is therefore

$$P_{0\to n} = \frac{1}{\hbar^2}\left|\int_{-\infty}^{\infty} e^{in\omega t}\,\langle n,0,0|H'|g\rangle\,\mathrm{d}t\right|^2.$$

The coupling vanishes to first-order unless $n = 1$, in which case

$$\langle 1,0,0|H'|g\rangle = qE_0 e^{-t/\tau}\,\Theta(t)\sqrt{\frac{\hbar}{2m\omega}},$$

and so

$$\begin{aligned} P_{0\to 1} &= \frac{q^2E_0^2}{2m\hbar\omega}\left|\int_0^{\infty} e^{-t/\tau}e^{i\omega t}\,\mathrm{d}t\right|^2 \\ &= \frac{q^2E_0^2}{2m\hbar\omega}\frac{1}{|-1/\tau + i\omega|^2} \\ &= \boxed{\frac{q^2E_0^2\tau^2}{2m\hbar\omega\,(1+\omega^2\tau^2)}.} \end{aligned}$$