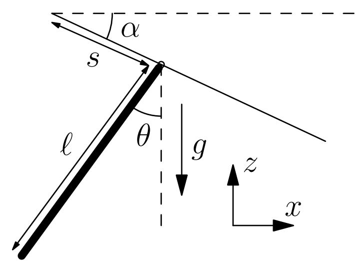
1 Dec 2021



# J10M.1 - Rod on a Rail (M93M.2)

## Problem



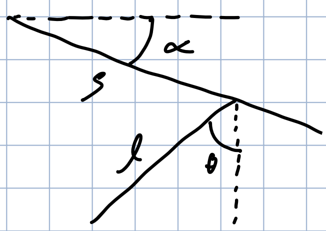

A uniform rod of length $\ell$ and mass $m$ moves in the $x$-$z$ plane. One end of the rod is suspended from a straight rail that slopes downwards with an angle $\alpha$ relative to the horizontal; the connection point is free to move along the rail without friction, and the rod is able to swing freely in the $x$-$z$ plane. Uniform gravity acts downwards.

a) Construct the Lagrangian of this system in terms of generalized coordinates $s$ (the distance the connection point has moved along the rail) and $\theta$ (the angle the rod makes with the vertical direction).

b) Using your Lagrangian, find a solution to the equation of motion where the rod moves with fixed $\theta$ as $s$ increases.

c) Explain how your solution is consistent with (and can be derived from) the equivalence principle.

J10M.1




$$x = s\cos\alpha - \frac{l}{2}\sin\theta$$
$$y = -\left(s\sin\alpha + \frac{l}{2}\cos\theta\right)$$

a) $L = T - V$

$$T = T_{tran} + T_{rot}$$

$$T_{tran} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$\dot{x} = \dot{s}\cos\alpha - \frac{l}{2}\dot{\theta}\cos\theta \quad \dot{y} = -\left(\dot{s}\sin\alpha - \frac{l}{2}\dot{\theta}\sin\theta\right)$$

$$\dot{x}^2 + \dot{y}^2 = \dot{s}^2 + \frac{1}{4}(l\dot{\theta})^2 - l\dot{\theta}\dot{s}(\cos\alpha\cos\theta + \sin\alpha\sin\theta)$$
$$= \dot{s}^2 + \frac{1}{4}(l\dot{\theta})^2 - l\dot{\theta}\dot{s}\cos(\theta - \alpha)$$

$$= \frac{1}{2}m\dot{s}^2 + \frac{1}{8}m(l\dot{\theta})^2 - \frac{1}{2}l\dot{\theta}\dot{s}\cos(\theta - \alpha)$$

$$T_{rot} = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{12}ml^2\right)\dot{\theta}^2 = \frac{1}{24}m(l\dot{\theta})^2$$

$$V = mgy = -mgs\sin\alpha - mg\frac{l}{2}\cos\theta$$

$$\boxed{L = \frac{1}{2}m\dot{s}^2 + \frac{1}{6}m(l\dot{\theta})^2 - \frac{1}{2}l\dot{\theta}\dot{s}\cos(\theta - \alpha) + mgs\sin\alpha + mg\frac{l}{2}\cos\theta}$$

b) $\dot{\theta} = \ddot{\theta} = 0$

$$\frac{\partial L}{\partial \theta} = -\frac{1}{2}l\dot{\theta}\dot{s}\sin(\theta - \alpha) - \frac{1}{2}mgl\sin\theta \longrightarrow -\frac{1}{2}mgl\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(\frac{1}{3}ml^2\dot{\theta} - \frac{1}{2}l\dot{s}\cos(\theta - \alpha)\right) = \frac{1}{2}ml^2\ddot{\theta} - \frac{1}{2}l\ddot{s}\cos(\theta - \alpha) + \frac{1}{2}l\dot{s}\dot{\theta}\sin(\theta - \alpha) \longrightarrow -\frac{1}{2}l\ddot{s}\cos(\theta - \alpha)$$

$$g\sin\theta = \ddot{s}\cos(\theta - \alpha)$$

$$\frac{\partial L}{\partial s} = mg\sin\alpha \rightarrow mg\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) = \frac{d}{dt}\left(m\dot{s} - \frac{1}{2}l\dot{\theta}\cos(\theta-\alpha)\right) = m\ddot{s} - \frac{1}{2}l\ddot{\theta}\cos(\theta-\alpha) + \frac{1}{2}l\dot{\theta}^2\sin(\theta-\alpha) \rightarrow m\ddot{s}$$

$$g\sin\theta = \ddot{s}$$

$$\cos(\theta-\alpha) = 1 \rightarrow \boxed{\theta = \alpha}$$

c) Frame w/ gravity = frame accelating at $g$

both have $\theta = \alpha \leftarrow$ just calculated for 1st case, trivial to second