Princeton Physics Prelims: J10E.3 Rectangular Waveguide

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(a) Find the corresponding $\vec{B}$

We are given $\vec{E} = E_x\hat{x} = E_0 \sin(\pi y/b)e^{i(kz-\omega t)}\hat{x}$ in the waveguide. A magnetic field in a wave propagating in the $\hat{z}$ direction has the general form $\vec{B} = B_0(x, y, z)e^{i(kz-\omega t)}$. Substitute these into the relevant Maxwell's equation in the waveguide: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$. We get that $\vec{B}$ has components in the y and z directions:

$$\left(\frac{\partial E_x}{\partial z}\right)\hat{y} + \left(-\frac{\partial E_x}{\partial y}\right)\hat{z} = -\frac{\partial \vec{B}}{\partial t} \tag{1}$$

$$\boxed{B_{0,y} = \frac{k}{\omega}E_0 \sin\left(\frac{\pi y}{b}\right), B_{0,z} = \frac{\pi}{b\omega}E_0 \cos\left(\frac{\pi y}{b}\right)} \tag{2}$$

$$\boxed{\vec{B} = \left(B_{0,y}\hat{y} + B_{0,z}\hat{z}\right)e^{i(kz-\omega t)}} \tag{3}$$

(b) Find the transmitted $\vec{E}$ in the newly created vacuum region

We assume the reflected (R) and transmitted (T) electric fields have the same general form. Note that the reflected wave has the reversed wavenumber $-k$ (since it is propagating in $-\hat{z}$), the reflected B-field has reversed direction compared to the incident magnetic field, and that the transmitted fields have a different wavenumber k_2.

$$\vec{E}_R = E_{0,R}\sin(\pi y/b)e^{i(-kz-\omega t)}\hat{x} \tag{4}$$

$$\vec{B}_R = \frac{1}{\omega/k}(\hat{k}_R \times \vec{E}_R) = \frac{1}{\omega/k}E_{0,R}\sin(\pi y/b)e^{i(-kz-\omega t)}(-\hat{y}) \tag{5}$$

$$\vec{E}_T = E_{0,T}\sin(\pi y/b)e^{i(k_2 z-\omega t)}\hat{x} \tag{6}$$

$$\vec{B}_T = \frac{1}{c}(\hat{k}_T \times \vec{E}_T) \tag{7}$$

The boundary conditions are $(E_1^{\parallel} - E_2^{\parallel})\big|_{z=0} = 0$ and $(B_1^{\parallel}/\mu_1 - B_2^{\parallel}/\mu_2)\big|_{z=0} = 0$.

The first boundary condition gives

$$E_0 + E_{0,R} = E_{0,T} \tag{8}$$

The second boundary condition, using $\mu_1 = \mu_2 = \mu_0$, and using the fact that the reflected magnetic field $\vec{B}_R$

changes sign, gives

$$B_{0,y} + B_R - B_T = 0 \tag{9}$$

$$\frac{k}{\omega}E_0 \sin(\pi y/b)\hat{y} + \frac{k}{\omega}E_{0,R}\sin(\pi y/b)(-\hat{y}) - \frac{1}{c}E_{0,T}\sin(\pi y/b)\hat{y} = 0 \tag{10}$$

$$\frac{k}{\omega}(E_0 - E_{0,R}) = \frac{1}{c}E_{0,T} \tag{11}$$

$$E_0 - E_{0,R} = \frac{\omega}{kc}E_{0,T} \tag{12}$$

Adding Eqn. (12) and Eqn. (8),

$$2E_0 = E_{0,T} + \frac{\omega}{kc}E_{0,T} \tag{13}$$

$$E_{0,T} = \frac{2}{1+\omega/(kc)}E_0 \tag{14}$$

We can find the transmitted field's wavenumber k_2 using the wave equation in the vacuum ($z > 0$),

$$\nabla^2\vec{E} = \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial^2 t} \tag{15}$$

$$-{k_y}^2 - k^2 = \frac{1}{c^2}(-i\omega)^2 \tag{16}$$

$$(\pi/b)^2 + {k_2}^2 = \frac{1}{c^2}\omega^2 \tag{17}$$

This gives the transmitted field's wavenumber (taking the positive root for propagating waves)

$$\boxed{k_2 = \sqrt{\frac{\omega^2}{c^2} - \left(\frac{\pi}{b}\right)^2}} \tag{18}$$

$$\boxed{\vec{E}_T = \frac{2E_0}{1+\omega/(kc)}\sin(\pi y/b)e^{i(k_2 z - \omega t)}\hat{x}} \tag{19}$$

(c) Range of ω that will lead to no transmitted propagating wave in $z > 0$

k_2 must have no imaginary component, or the transmitted wave will have a decaying term $e^{-\alpha z}$ for some real α.

$$\frac{\omega^2}{c^2} - \left(\frac{\pi}{b}\right)^2 > 0 \tag{20}$$

$$\boxed{\omega > c\frac{\pi}{b}} \tag{21}$$