J10E.2 - Rotating Sphere in a Magnetic Field (Solution by Jim Wu)

A solid metallic sphere of radius a has finite conductivity, carries no net electric charge, and is free to rotate without friction about a vertical axis through its center. The region outside the sphere is vacuum. There is a uniform magnetic field with flux density $\vec{B}_0$ parallel to the axis.
The sphere is given an impulse that starts it spinning around the axis and there is some initial Ohmic dissipation. After the dissipation has ceased, the sphere is in a steady state of rigid rotation with constant angular velocity ω_∞.
In steady state, to *lowest* order in both B_0 and ω_∞, find:

(a) The electric field $\vec{E}(\vec{r})$ and electric potential $\Phi(\vec{r})$ in the interior of the sphere $r < a$. (Give these in the non-rotating "laboratory frame").

(b) The electric potential outside the sphere. (Express your answer in spherical coordinates (r, θ, ϕ).) State the nature of the electric field it describes (i.e., monopole, dipole, quadrupole, etc.)

(c) The induced bulk and surface charge density distributions in the conductor that give rise to this potential.

Solution:

(a) Since we do not have any driving electromotive forces, then the charges in the sphere of finite conductivity cannot be moving relative to the sphere. This suggests that the charges do not feel an electromagnetic forces. Hence,

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} = -(\boldsymbol{\omega} \times \mathbf{r}) \times \mathbf{B} = B_0 \omega_\infty r \hat{\mathbf{z}} \times (\hat{\mathbf{z}} \times \hat{\mathbf{r}}) = B_0 \omega_\infty r (\hat{\mathbf{z}}(\hat{\mathbf{z}} \cdot \hat{\mathbf{r}}) - \hat{\mathbf{r}})$$

In spherical coordinates, $\hat{\mathbf{z}} = \cos\theta \hat{\mathbf{r}} - \sin\theta \hat{\boldsymbol{\theta}}$. So, the electric field inside the sphere is

$$\mathbf{E} = B_0 \omega_\infty r \left(\cos\theta (\cos\theta \hat{\mathbf{r}} - \sin\theta \hat{\boldsymbol{\theta}}) - \hat{\mathbf{r}} \right) = \boxed{-B_0 \omega_\infty r (\sin^2\theta \hat{\mathbf{r}} + \sin\theta \cos\theta \hat{\boldsymbol{\theta}})}$$

The relation between the electric potential and the electric field is

$$\mathbf{E} = -\nabla\Phi = -\left(\frac{\partial\Phi}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial\Phi}{\partial\theta} \hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta} \frac{\partial\Phi}{\phi} \hat{\boldsymbol{\phi}} \right)$$

Since the electric field has no component in the $\hat{\boldsymbol{\phi}}$ components, we are left with the equations

$$\frac{\partial\Phi}{\partial r} = B_0 \omega_\infty r \sin^2\theta \qquad \frac{\partial\Phi}{\partial\theta} = B_0 \omega_\infty r^2 \sin\theta \cos\theta$$

Solving either equation yields the potential inside the sphere:

$$\Phi = \Phi_0 + \frac{1}{2} B_0 \omega_\infty r^2 \sin^2\theta$$

where Φ_0 is some constant. Note that we can express the potential in terms of the Legendre polynomials $P_0(\cos\theta) = 1$ and $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$:

$$\begin{aligned}\Phi_{in} &= \Phi_0 + \frac{1}{2}B_0\omega_\infty r^2(1-\cos^2\theta) \\ &= \Phi_0 + \frac{1}{2}B_0\omega_\infty r^2\left(\frac{2}{3}P_0 - \frac{2}{3}P_2\right) \\ &= \left(\Phi_0 + \frac{B_0\omega_\infty r^2}{3}\right)P_0 - \frac{B_0\omega_\infty r^2}{3}P_2\end{aligned}$$

(b) From solving Laplace's equation, we can tell that the potential outside must be of the form

$$\Phi_{out} = \sum_\ell \frac{A_\ell}{r^{\ell+1}}P_\ell(\cos\theta)$$

However, to ensure continuity of the electric potential at the boundary, we can only have $\ell = 0, 2$ and so

$$\Phi_{out} = \frac{A_0}{r}P_0(\cos\theta) + \frac{A_2}{r^3}P_2(\cos\theta)$$

At $r = a$, we find that

$$A_0 = \left(\Phi_0 + \frac{B_0\omega_\infty a^2}{3}\right)a \qquad\qquad A_2 = -\frac{B_0\omega_\infty a^5}{3}$$

Rewriting in terms of θ, the potential outside is

$$\Phi_{out} = \Phi_0\frac{a}{r} + \frac{B_0\omega_\infty a^3}{3r} - \frac{B_0\omega_\infty a^5}{3r^3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

The electric field outside the sphere is

$$\begin{aligned}\mathbf{E}_{out} &= -\nabla\Phi \\ &= -\frac{\partial\Phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\boldsymbol{\theta}} \\ &= \left(\left(\Phi_0 a + \frac{B_0\omega_\infty a^3}{3}\right)\frac{1}{r^2} - \frac{B_0\omega_\infty a^5}{2r^4}(3\cos^2\theta - 1)\right)\hat{\mathbf{r}} - \frac{B_0\omega_\infty a^5}{r^4}\cos\theta\sin\theta\hat{\boldsymbol{\theta}}\end{aligned}$$

From the $\frac{1}{r^2}$ and $\frac{1}{r^4}$ dependencies, we see that the electric field outside has some monopole and quadrupole character. Notice that $\mathbf{E}^\theta_{in}(a) = \mathbf{E}^\theta_{out}(a)$, as expected.

On the other hand, the normal component of the field at the boundary is discontinuous as there is a charge density on the surface of

$$\begin{aligned}\sigma &= \epsilon_0(E^r_{out}(r=a) - E^r_{in}(r=a)) \\ &= \epsilon_0\left[\left(\frac{\Phi_0}{a} + \frac{B_0\omega_\infty a}{3} - \frac{B_0\omega_\infty a}{2}(3\cos^2\theta - 1)\right) - \left(-B_0\omega_\infty a(1-\cos^2\theta)\right)\right] \\ &= \epsilon_0\left[\frac{\Phi_0}{a} + \frac{11B_0\omega_\infty a}{6} - \frac{5B_0\omega_\infty a}{2}\cos^2\theta\right]\end{aligned}$$

To find the value of Φ_0, we must use the condition that the sphere is electrically neutral, meaning that the total surface charge and total charge inside the sphere must cancel entirely. Using the surface charge density derived above, the total surface charge of this sphere is

$$\begin{aligned} Q_S &= \int_0^\pi \sigma 2\pi a^2 \sin\theta \, d\theta \\ &= \epsilon_0 \left[\left(\frac{\Phi_0}{a} + \frac{11 B_0 \omega_\infty a}{6} \right) 4\pi a^2 - \frac{5 B_0 \omega_\infty a}{2} (2\pi a^2) \int_0^\pi \sin\theta \cos^2\theta \, d\theta \right] \\ &= \epsilon_0 \left[\left(\frac{\Phi_0}{a} + \frac{11 B_0 \omega_\infty a}{6} \right) 4\pi a^2 - \frac{5 B_0 \omega_\infty a}{2} (2\pi a^2) \left(\frac{2}{3} \right) \right] \\ &= 4\pi \epsilon_0 \Phi_0 a + 4 B_0 \pi \omega_\infty a^3 \end{aligned}$$

The volume charge density inside the sphere can be found via Gauss' law:

$$\begin{aligned} \rho &= \epsilon_0 \nabla \cdot \mathbf{E}_{in} \\ &= \epsilon_0 \left(\frac{1}{r^2} \partial_r (r^2 E_r) + \frac{1}{r \sin\theta} \partial_\theta (\sin\theta E_\theta) \right) \\ &= \epsilon_0 \left(-3 B_0 \omega_\infty \sin^2\theta - B_0 \omega_\infty (2\cos^2\theta - \sin^2\theta) \right) \\ &= -B_0 \omega_\infty \epsilon_0 (3\sin^2\theta + 2\cos^2\theta - \sin^2\theta) \\ &= -2 B_0 \omega_\infty \epsilon_0 \end{aligned}$$

Since ρ is constant, the total charge inside the sphere is

$$Q_V = \rho \left(\frac{4}{3} \pi a^3 \right) = -\frac{8}{3} \pi B_0 \epsilon_0 \omega_\infty a^3$$

Using the fact that $Q_S + Q_V = 0$, the constant Φ_0 is

$$4\pi \epsilon_0 \Phi_0 a + 4 B_0 \pi \omega_\infty a^3 - \frac{8}{3} \pi B_0 \epsilon_0 \omega_\infty a^3 = 0 \qquad \Rightarrow \qquad \Phi_0 = -\frac{1}{3} B_0 \omega_\infty a^2$$

giving us the full expression for the potential inside and outside the sphere.

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