Problem J10E.1

(a) Since $d \ll \lambda_0$, the field is described to a good approximation by an image dipole

$$\mathbf{p}'(t) := -\mathbf{p}(t)$$

located a distance d below the plane. Thus the net electric dipole moment is zero in the far-field, and the dominant contribution is electric quadrupole radiation.

Let $\hat{\mathbf{z}}$ point out of the page and $\hat{\mathbf{x}}$ point right, so that the dipole is located at $d\,\hat{\mathbf{y}}$. Then the corresponding quadrupole tensor is

$$\begin{aligned}\mathbf{Q} &= 2d\,(\hat{\mathbf{y}} \otimes \mathbf{p}(t) + \mathbf{p}(t) \otimes \hat{\mathbf{z}}) \\ &= 2d\,(\hat{\mathbf{y}} \otimes \hat{\mathbf{x}} + \hat{\mathbf{x}} \otimes \hat{\mathbf{y}}) p_0 \cos(\omega t).\end{aligned}$$

The electric quadrupole contribution to the vector potential is

$$\mathbf{A}(t) = -\frac{1}{6c}\frac{\mu_0}{4\pi r}\hat{\mathbf{r}} \cdot \ddot{\mathbf{Q}}(t - r/c),$$

which evaluates to

$$\mathbf{A}(t) = \frac{dp_0\mu_0\omega^2}{12\pi r^2 c}(y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}})\,\cos(\omega(t - r/c)).$$

The corresponding magnetic field is (in the far-field limit) is

$$\begin{aligned}\mathbf{B}(t) &= -\hat{\mathbf{r}} \times \mathbf{A}(t) \\ &= -\frac{dp_0\mu_0\omega^2}{12\pi r^3 c}(-xz\,\hat{\mathbf{x}} + yz\,\hat{\mathbf{y}} + (x^2 - y^2)\,\hat{\mathbf{z}})\,\cos(\omega(t - r/c)) \\ &= -\frac{dp_0\mu_0\omega^2}{12\pi r c}(\cos\theta\sin\theta(-\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}) + \sin^2\theta\cos(2\phi)\,\hat{\mathbf{z}})\,\cos(\omega(t - r/c)),\end{aligned}$$

and the corresponding electric field is

$$\mathbf{E}(t) = -c\,\hat{\mathbf{r}} \times \mathbf{B}(t).$$

(b) The Poynting vector is proportional to $\mathbf{B}^2$ in the far-field, and so the radiated power follows the distribution

$$\begin{aligned}P(\theta, \phi) &\propto \sin^4\theta\cos^2(2\phi) + \cos^2\theta\sin^2\theta \\ &= \sin^2\theta\,(\cos^2\theta + \sin^2\theta\cos^2 2\phi).\end{aligned}$$

In particular, no power is radiated directly out of the page (at quadrupole order), and there are four 'lobes' of maximum emission within the xy plane aligned with the x and y axes.