Prelims Solutions

Problem J09T2

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1

The Boltzmann weight for m defects on the lattice is $e^{-m\Delta\beta}$ and the number of such microstates is $\binom{N}{m}$. So by definition $Z = 1 + Ne^{-\Delta\beta} + \binom{N}{2}e^{-2\Delta\beta} + ... = (1 + e^{-\Delta\beta})^N$.

2

$< n >= 0 * 1/Z + 1 * Ne^{-\Delta\beta}/Z + 2 * \binom{N}{2}e^{-2\Delta\beta}/Z + ... = \frac{U}{\Delta} = -\frac{1}{\Delta}\frac{\partial lnZ}{\partial\beta} = N\frac{1}{e^{\Delta\beta}+1}$.
And by definition $F = -\frac{1}{\beta}lnZ = -\frac{N}{\beta}ln(1 + e^{-\Delta\beta})$.

3

From thermodynamics $F = U - TS$ so $S(T) = \beta(-F + U) = Nln(1 + e^{-\Delta\beta}) + N\Delta\beta\frac{1}{e^{\Delta\beta}+1}$.
By definition $C_V = (\frac{\partial U}{\partial T})_V = \frac{N\Delta^2\beta^2e^{\Delta\beta}}{(e^{\Delta\beta}+1)^2}$.

4

W_n is just the number of ways to place n defects on N sites, $\binom{N}{n}$. For large N, $S(T) = ln(W_{<n>}) = Nln(N) - < n > ln(< n >) - (N - < n >)ln(N - < n >) = N[lnN - \frac{1}{(e^{\Delta\beta}+1)}ln(\frac{N}{(e^{\Delta\beta}+1)}) - \frac{e^{\Delta\beta}}{(e^{\Delta\beta}+1)}ln(\frac{Ne^{\Delta\beta}}{(e^{\Delta\beta}+1)})]$.
A bit of algebra indeed gives $S(T) = Nln(1 + e^{-\Delta\beta}) + N\Delta\beta\frac{1}{e^{\Delta\beta}+1}$ as before.

5

For low temperatures ($\Delta\beta >> 1$) most of the average energy U is likely to come from only a few defects somewhere in the lattice (because the probability of a defect $e^{-\Delta\beta}$ is very low). Hence $U \approx energy\ of\ defect * number\ of\ defects = \Delta * Ne^{-\Delta\beta}$. The heat capacity is the T derivative of U $C_V \approx N\Delta^2\beta^2e^{-\Delta\beta}$, which agrees with the C_V of part (c) in the low T limit.