1

The Boltzmann weight for \( m \) defects on the lattice is \( e^{-m\Delta\beta} \) and the number of such microstates is \( \binom{N}{m} \). So by definition
\[
Z = 1 + Ne^{-\Delta\beta} + \binom{N}{2} e^{-2\Delta\beta} + \ldots = (1 + e^{-\Delta\beta})^N.
\]

2

\[
\langle n \rangle = 0 \times \frac{1}{Z} + 1 \times \frac{Ne^{-\Delta\beta}}{Z} + 2 \times \frac{\binom{N}{2} e^{-2\Delta\beta}}{Z} + \ldots = \frac{U}{\Delta} = -\frac{1}{\Delta} \frac{\partial \ln Z}{\partial \beta} = N \frac{1}{e^{\Delta\beta} + 1}.
\]

And by definition
\[
F = -\frac{1}{\beta} \ln Z = -\frac{N}{\beta} \ln (1 + e^{-\Delta\beta}).
\]

3

From thermodynamics \( F = U - TS \) so
\[
S(\Delta\beta) = \beta(-F + U) = N \ln (1 + e^{-\Delta\beta}) + N\Delta\beta \frac{1}{e^{\Delta\beta} + 1}.
\]

By definition
\[
C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{N\Delta^2 \beta^2 e^{\Delta\beta}}{(e^{\Delta\beta} + 1)^2}.
\]

4

\( W_n \) is just the number of ways to place \( n \) defects on \( N \) sites, \( \binom{N}{n} \). For large \( N \),
\[
S(T) = \ln(W_{<n>}) = N \ln(N) - \langle n \rangle - \ln(\langle n \rangle) - (N - \langle n \rangle) \ln(N - \langle n \rangle) = N \left\{ \ln \frac{N}{(e^{\Delta\beta} + 1)} - \frac{\Delta\beta}{(e^{\Delta\beta} + 1)} \ln \left( \frac{N e^{\Delta\beta}}{(e^{\Delta\beta} + 1)} \right) \right\}.
\]

A bit of algebra indeed gives
\[
S(T) = N \ln (1 + e^{-\Delta\beta}) + N \Delta\beta \frac{1}{e^{\Delta\beta} + 1}
\]
as before.

5

For low temperatures \((\Delta\beta >> 1)\) most of the average energy \( U \) is likely to come from only a few defects somewhere in the lattice (because the probability of a defect \( e^{-\Delta\beta} \) is very low). Hence
\[
U \approx \text{energy of defects} \times \text{number of defects} = \Delta \times Ne^{-\Delta\beta}.
\]
The heat capacity is the \( T \) derivative of \( U 
\[
C_V \approx N\Delta^2 \beta^2 e^{-\Delta\beta},
\]
which agrees with the \( C_V \) of part (c) in the low \( T \) limit.