j09t3

a. $P_n = \binom{N}{n} p^n (1-p)^{N-n}$

b. $p \ll 1$, $n \ll N$, $1 \ll N$

$$P_n = \frac{N!}{(N-n)!\,n!} p^n (1-p)^{N-n}$$

$$= \frac{p^n}{n!} \frac{N^N e^{-N}}{(N-n)^{N-n} e^{-(N-n)}} e^{-p(N-n)}$$

$$= \frac{p^n e^{-pN}}{n!} \frac{N^N}{(N-n)^N (N-n)^{-n} e^n} e^{np}$$

note $(N-n)^N = N^N (1-\frac{n}{N})^N \cong N^N(1-n) \approx N^N e^{-n}$

$$= \frac{p^n e^{-pN}}{n!} (N-n)^n e^{np}$$

note $(N-n)^n = N^n (1-\frac{n}{N})^n = N^n (1-\frac{n^2}{N})$

$$= \frac{p^n N^n e^{-pN}}{n!} (1-\frac{n^2}{N})(1+np) \approx \frac{p^n N^N e^{-pN}}{n!} = \frac{\lambda^n e^{-\lambda}}{n!}$$

c. $$\bar{n} = \sum \frac{n\lambda^n}{n!} e^{-\lambda} = e^{-\lambda} \sum_n \frac{\lambda \frac{\partial}{\partial\lambda} \lambda^n}{n!} = \lambda e^{-\lambda} \frac{\partial}{\partial\lambda} \sum \frac{\lambda^n}{n!}$$

$$= \lambda e^{-\lambda} \frac{\partial}{\partial\lambda}(e^\lambda) = \lambda.$$

$$\overline{n^2} = \sum \frac{n^2 \lambda^n e^{-\lambda}}{n!} = e^{-\lambda} \sum \frac{\lambda \frac{\partial}{\partial\lambda}(n\lambda^n)}{n!} = \lambda e^{-\lambda} \frac{\partial}{\partial\lambda} \sum \frac{n\lambda^n}{n!}$$

note from above $\lambda = \bar{n} = e^{-\lambda} \sum \frac{n\lambda^n}{n!} \to \sum \frac{n\lambda^n}{n!} = \lambda e^\lambda$

so $\overline{n^2} = \lambda e^{-\lambda} \frac{\partial}{\partial\lambda}(\lambda e^\lambda) = \lambda e^{-\lambda}(e^\lambda + \lambda e^\lambda) = \lambda(\lambda+1)$.

$$\overline{(n-\bar{n})^2} = \overline{n^2 - n\bar{n} + \bar{n}^2} = \overline{n^2} - \bar{n}^2 = \lambda(\lambda+1) - \lambda^2 = \lambda.$$

d. $p = 10^{-6}$, $N = 3\times10^4 \to \lambda = Np = .03$, $e^{-\lambda} \cong 1-\lambda = .97$

$P_0 = .97$

$P_1 = (.03)(.97) = .0291$

$P_2 = \frac{(.03)^2 \cdot .97}{2} = .0004365$