joint

a. $\langle E \rangle = \frac{1}{Z} \sum E_s e^{-\beta E_s} = \frac{1}{Z} \sum -\frac{\partial}{\partial \beta} e^{-\beta E_s} = \frac{1}{Z} \left(-\frac{\partial Z}{\partial \beta} \right)$

$= -\frac{\partial \log Z}{\partial \beta}$.

b. $\langle P \rangle = -\left\langle \frac{\partial E_s}{\partial V} \right\rangle = \frac{-1}{Z} \sum \frac{\partial E_s}{\partial V} e^{-\beta E_s} = -\frac{1}{Z\beta} \sum \frac{\partial}{\partial V} (E_s \beta) e^{-\beta E_s}$

$= \frac{-1}{Z\beta} \frac{\partial}{\partial V} Z = -\frac{1}{\beta} \frac{\partial \log Z}{\partial \beta}$

c. $F = E - TS$

$S = -k \sum \log \left(\frac{e^{-\beta E_s}}{Z} \right) \frac{e^{-\beta E_s}}{Z}$

$= \frac{-k}{Z} \sum \log (-\beta E_s) e^{-\beta E_s} - \log Z e^{-\beta E_s}$

$S = k\beta \langle E \rangle + k \log Z$

$F = \langle E \rangle - kT\beta \langle E \rangle - kT \log Z \rightarrow F = -kT \log Z$.

d. $\left\langle (E - \bar{E})^2 \right\rangle = \langle E^2 - 2E\bar{E} + \bar{E}^2 \rangle = \overline{E^2} - \bar{E}^2$.

$\bar{E} = -\frac{\partial \log Z}{\partial \beta}$ consider $\frac{\partial^2 \log Z}{\partial \beta^2} = \frac{\partial}{\partial \beta} \left(\frac{1}{Z} \frac{\partial Z}{\partial \beta} \right) = \left(-\frac{1}{Z^2} \left(\frac{\partial Z}{\partial \beta} \right)^2 + \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2} \right)$

and $\overline{E^2} = \frac{1}{Z} \sum E^2 e^{-\beta E_s}$ $\qquad \downarrow -\bar{E}^2 \qquad \frac{1}{E^2}$

$= \frac{1}{Z} \sum \frac{\partial^2}{\partial \beta^2} e^{-\beta E_s}$ $\qquad$ so $\sigma_E = \frac{\partial^2 \log Z}{\partial \beta^2}$.

$= \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2}$