Jan 2009 #2 (QM)

$$\vec{B} = \begin{cases} 0 & x<0 \\ B_0 \hat{z} & x \geq 0 \end{cases}$$

electron with spin in $\hat{z}$ is incident from $x<0$, with velocity $v\hat{x}$

$$H = \frac{\left(\vec{p} - \frac{q\vec{A}}{c}\right)^2}{2m} + \frac{eB}{mc} S_z$$

Choose gauge where $\hat{A} = Bx\hat{y}$

Then $$H = \begin{cases} \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} - \frac{qB}{mc} x p_y + \frac{q^2 B^2}{2mc^2} x^2 - \frac{qB}{mc} S_z & x \geq 0 \\ \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} & x<0 \end{cases}$$

- $\frac{p_z^2}{2m}$ is irrelevant to the motion; remove it from problem
- Also, $p_y$ commutes with the Hamiltonian, so $p_y \to \hbar k_y$ (eigenvalue) with $e^{ik_y y}$ eigenfunction
- $S_z$ commutes, so $S_z \to \frac{\hbar}{2}$

for $x \geq 0$, $H = \frac{p_x^2}{2m} + \frac{\hbar^2 k_y^2}{2m} - \frac{qB}{mc} x \hbar k_y + \frac{q^2 B^2}{2mc^2} x^2 - \frac{qB\hbar}{2mc}$

$$H = \frac{p_x^2}{2m} + \frac{1}{2} m\omega^2 (x - x_0)^2 - \frac{qB\hbar}{2mc} \qquad \text{where } \omega \equiv \frac{qB}{mc} \quad x_0 = \frac{\hbar k_y}{m\omega}$$

$x<0$: $H\psi = E\psi \Rightarrow E = \frac{\hbar^2 k_x^2}{2m} + \frac{\hbar^2 k_y^2}{2m} = \frac{1}{2} m v^2 + \frac{\hbar^2 k_y^2}{2m}$

$\frac{p_x^2}{2m}\psi + \frac{\hbar^2 k_y^2}{2m}\psi = E\psi$ (effective potential of $\frac{\hbar^2 k_y^2}{2m}$)

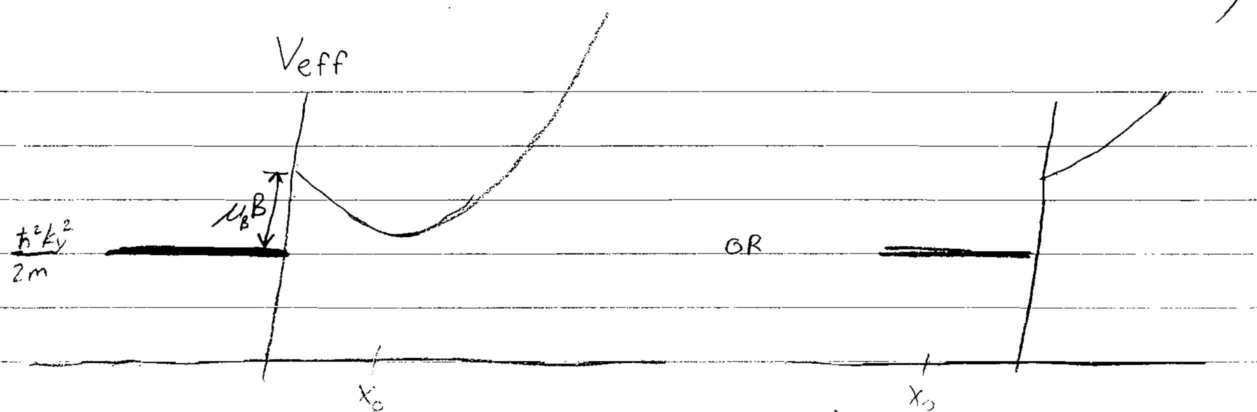
$x>0$: $H\psi = E\psi$:

$$\frac{p_x^2 \psi}{2m} + \frac{1}{2} m\omega^2 (x - x_0)^2 \psi - \left(\frac{qB\hbar}{2mc} + E\right)\psi = 0$$

effective potential of $\frac{1}{2} m\omega^2 (x - x_0)^2 - \frac{qB\hbar}{2mc}$

at $x=0$, $V_{eff} = \frac{1}{2} m\omega^2 \cdot \frac{\hbar^2 k_y^2}{m^2 \omega^2} - \frac{qB\hbar}{2mc}$

$$= \frac{\hbar^2 k_y^2}{2m} - \frac{qB\hbar}{2mc} = \frac{\hbar^2 k_y^2}{2m} + \frac{e\hbar}{2mc} B = \frac{\hbar^2 k_y^2}{2mc} + \mu_B B$$




(depends on sign of $x_0$ = sign of $k_y$)

$E = \frac{\hbar^2 k_y^2}{2m} + \frac{1}{2} m v^2$

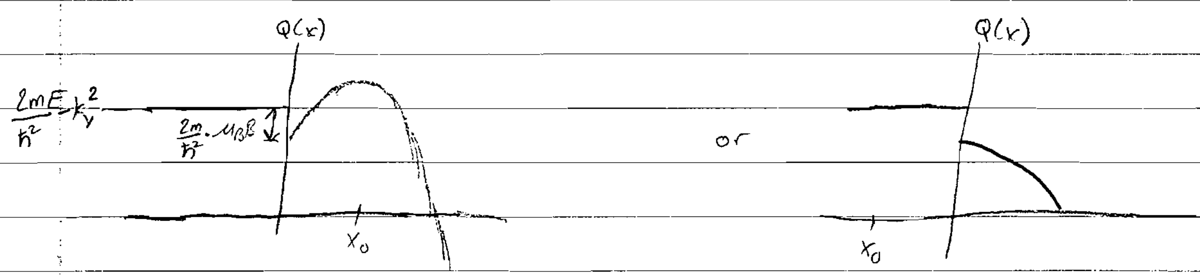
To behave semi-classically, the wavefunction should not decay immediately, and $E - V_{eff}$ should be approximately continuous, not near 0

Therefore, $\frac{1}{2} m v^2 >> \mu_B B = \frac{e\hbar B}{2m_e c}$

For $x>0$, $\frac{d^2\psi}{dx^2} + \left[ \frac{2m}{\hbar^2}\left(E - \mu_B B\right) - \frac{m^2\omega^2}{\hbar^2}\left(x - x_0\right)^2 \right]\psi = 0$

[Solution method from Asymptotic Analysis course]

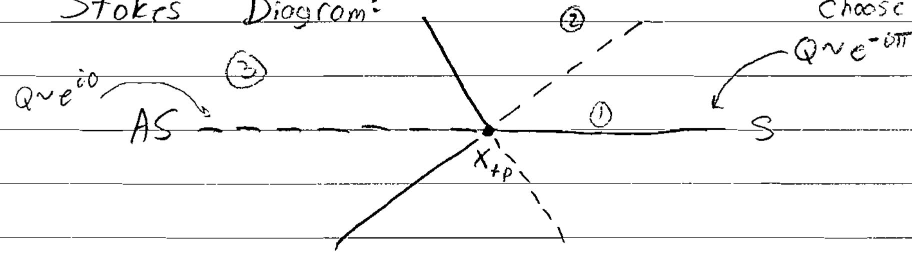
$Q(x) = \frac{2m}{\hbar^2}\left(E - \mu_B B\right) - \frac{m^2\omega^2}{\hbar^2}\left(x - x_0\right)^2 \qquad x>0$




There is one 0, or turning point, of $Q(x)$, which occurs at

$x_{tp} = x_0 + \sqrt{\frac{2(E - \mu_B B)}{m\omega^2}}$ To the left: oscillating
To the right: decaying solution

Stokes Diagram:



WKB solutions are $\psi = \frac{1}{Q^{1/4}} e^{\pm i\int^x \sqrt{Q}\,dx}$

Region ①: Q has phase $e^{-i\pi} \rightarrow \sqrt{Q} \sim -i$

$(x_{tp}, x) \sim e^{\int_{x_{tp}}^{x} \sqrt{|Q|}dx}$ exponentially growing — dominant

$(x, x_{tp}) \sim e^{-\int_{x_{tp}}^{x} \sqrt{|Q|}dx}$ exponentially decreasing — subdominant

• Normalizable solution → solution must be decreasing

solution is $(x, x_{tp})_s$

① $(x, x_{tp})_s$

② $(x, x_{tp})_d$

③ $(x, x_{tp})_d + i(x_{tp}, x)_s$  $i$ = Stokes Constant for an isolated zero

on AS line solution is $\psi = (x, x_{tp}) + i(x_{tp}, x)$

$$\boxed{\psi = \frac{e^{ik_y y}}{Q^{1/4}}\left[e^{i\int_x^{x_{tp}}\sqrt{Q}\,dx} + i e^{-i\int_x^{x_{tp}}\sqrt{Q}\,dx}\right]}$$ with $e^{ik_y y}$ added in

Reflected  incident

For $x<0$, this gives $\psi = e^{-ik_x x} + i e^{ik_x x}$

With $\frac{1}{2}mv^2 >> \mu_B B$

b. ~~velocity:~~ $\frac{1}{m}\langle\psi|p_x|\psi\rangle$ and $\frac{1}{m}\langle\psi|p_y|\psi\rangle$ ?

$\cancel{\langle\psi|p_y|\psi\rangle \sim \hbar k_y \psi\psi^* \sim (e^{ia}+ie^{-ia})(e^{-ia}-ie^{ia}) \sim 2+2\sin(2a)}$

$\psi^* p_x \psi \sim \frac{1}{i}\psi^*\frac{\partial}{\partial x}\psi \sim (e^{-ia}-ie^{ia}) \cdot \frac{[(-i\sqrt{Q})e^{ia}+i(i\sqrt{Q})e^{-ia}]}{i}$

$\cancel{\sim (e^{-ia}-ie^{ia})(-e^{ia}+ie^{-ia})}$

$\sim -1 + ie^{-2ia} + ie^{2ia} + 1 \sim 2i\cos(2a)$

$V_x \sim$