j09m2

a. Want to minimize $D = a\int_0^L r(z)\left(\frac{dr}{dz}\right)^3 dz$

s.t. $r(0)=0$, $r(L)=d$.

Beltrami Identity : $r\dot{r}^3 - 3r\dot{r}^3 = c$

$\rightarrow r\dot{r}^3 = c$

$$\frac{dr}{dz} = \left(\frac{c}{r}\right)^{1/3}$$

$$\int dr\, c\, r^{1/3} = dz$$

$$c r^{4/3} = z \rightarrow r(z) = c z^{3/4}$$

satisfies $r(0)=0$. $r(L) = cL^{3/4} = d \rightarrow c = \frac{d}{L^{3/4}}$

so $r(z) = d\left(\frac{z}{L}\right)^{3/4}$

b. Now maximize D s.t. $V = \int_0^L \pi r^2 dz$ V a fixed volume.

Lagrange multipliers: extremize

$$D + \lambda V = \int_0^L \left(a r\dot{r}^3 + \lambda\pi r^2\right) dz$$

Beltrami Identity works again!

$$a r\dot{r}^3 + \lambda\pi r^2 - 3ar\dot{r}^3 = c$$

$$-2ar\dot{r}^3 + \lambda\pi r^2 = c$$

this is a first order ode that optimizing $r(z)$ will satisfy!