Jan 2009 #3 (EM)

$J_z = J(r) \quad B_\phi = B(r)$: z-pinch

total current I

a. $\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} \qquad \oint \vec{B}\cdot d\vec{l} = \frac{4\pi}{c}\int \vec{J}\cdot d\vec{a}$

$$2\pi r B_\phi = \frac{4\pi}{c} I \quad \text{for } r \geq R$$

$$B_\phi = \frac{2I}{rc}, \quad r \geq R$$

b. constant current density: $J = \frac{I}{\pi R^2}$

$$2\pi r B_\phi = \frac{4\pi}{c}\cdot\frac{I}{\pi R^2}\int_0^r r' dr' d\phi$$

$$\cancel{2\pi} r B_\phi = \frac{4I}{cR^2}\cdot 2\pi\cdot\frac{1}{2}r^2$$

$$B_\phi = \frac{2Ir}{cR^2} \qquad r \leq R$$

c. $\frac{\vec{J}\times\vec{B}}{c} = \nabla P = \frac{dP}{dr}\hat{r} \qquad -\frac{1}{c}J_z B_\phi = \frac{dP}{dr}$ Force Balance

$(\nabla\times\vec{B})_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\phi) \qquad \nabla\times\vec{B} = \frac{4\pi}{c}\vec{J}$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}(rB_\phi) = \frac{4\pi}{c}J_z \qquad \frac{J_z}{c} = \frac{1}{4\pi}\frac{1}{r}\frac{\partial}{\partial r}(rB_\phi)$$

$$\frac{dP}{dr} = -\frac{1}{4\pi}\frac{B_\phi}{r}\frac{\partial}{\partial r}(rB_\phi)$$

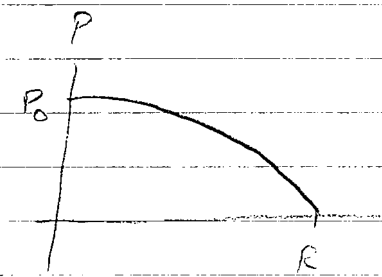
$$B_\phi = \left(\frac{2I}{cR^2}\right)r$$

$$\frac{dP}{dr} = -\frac{1}{4\pi}\cdot\left(\frac{2I}{cR^2}\right)^2\cdot\frac{\partial}{\partial r}(r^2) = -\frac{1}{2\pi}\left(\frac{2I}{cR^2}\right)^2 r$$

$$P(r) = P_0 - \frac{1}{4\pi}\left(\frac{2I}{cR^2}\right)^2 r^2 = -\frac{I^2}{\pi c^2 R^4}r^2 + P_0$$

$P(R)=0 \Rightarrow \quad P_0 = \dfrac{I^2}{\pi c^2 R^2}$

$$P(r) = \frac{I^2}{\pi c^2 R^2}\left(1 - \frac{r^2}{R^2}\right)$$



d. current in a very thin layer: $J_z = \delta(r-R)\dfrac{I}{2\pi R}$

Thus $\displaystyle\int \vec{J}\cdot d\vec{a} = \int r\,dr\,d\phi\, \frac{I}{2\pi R}\,\delta(r-R) = I\int \frac{r\,dr}{R}\,\delta(r-R) = I$

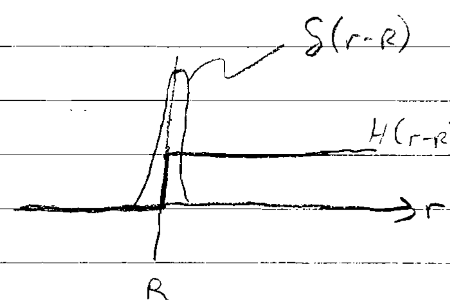
and $B_\phi = \begin{cases} \frac{2I}{rc} & r \geq R \\ 0 & r < R \end{cases} \Rightarrow B_\phi = \dfrac{2I}{rc}\,H(r-R)$ Heaviside step function

$$\frac{dP}{dr} = -\frac{J_z B_\phi}{c} = \frac{-I}{2\pi R c}\cdot\frac{2I}{rc}\,\delta(r-R)\,H(r-R)$$

$$\frac{dP}{dr} = \frac{-I^2}{\pi c^2 r R}\,\delta(r-R)\,H(r-R)$$

$$P(r) - P_0 = \frac{-I^2}{\pi c^2 R}\int_0^r dr'\,\frac{\delta(r'-R)H(r'-R)}{r'}$$

$$\rightarrow \frac{-I^2}{\pi c^2 R^2}\underbrace{\int_0^r dr'\,\delta(r'-R)\,H(r'-R)}_{\text{This integral is } \frac{1}{2}, \text{ if } r>R \text{ and } 0, \text{ if } r<R}$$



$$P = \begin{cases} P_0 & r<R \\ P_0 - \dfrac{I^2}{2\pi c^2 R^2} & r>R \end{cases}$$ and $P(r>R) = 0$, so $\boxed{P_0 = \dfrac{I^2}{2\pi c^2 R^2}}$