Jan 2009 #1 (EM)

Uniform magnetic field B in z-direction

uniform electric field E in x-direction

particle of mass m, charge q at origin

$$m\frac{dU^\alpha}{d\tau} = qF^\alpha_{\ \beta}U^\beta \qquad (c=1)$$

Assuming $B^2 > E^2$

- Field tensor

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

Antisymmetric

- Boost into a frame moving at speed $\frac{E}{B}$ in the $-y$ direction

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E & 0 & 0 \\ E & 0 & -B & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \Lambda^\alpha_\beta = \begin{bmatrix} \gamma & 0 & \gamma\frac{E}{B} & 0 \\ 0 & 1 & 0 & 0 \\ \gamma\frac{E}{B} & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To find fields in the new frame, transform the field tensor $F^{\alpha\beta}$ with the Lorentz Transformation matrix Λ

$$F^{\alpha'\beta'} = \Lambda^{T\alpha'}_{\ \ \alpha} F^{\alpha\beta} \Lambda^{\beta'}_\beta$$

$$\Rightarrow \begin{bmatrix} 0 & -E_x' & -E_y' & -E_z' \\ E_x' & 0 & -B_z' & B_y' \\ E_y' & B_z' & 0 & -B_x' \\ E_z' & -B_y' & B_x' & 0 \end{bmatrix} = \begin{bmatrix} \gamma & 0 & \gamma\frac{E}{B} & 0 \\ 0 & 1 & 0 & 0 \\ \gamma\frac{E}{B} & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -E & 0 & 0 \\ E & 0 & -B & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma & 0 & \gamma\frac{E}{B} & 0 \\ 0 & 1 & 0 & 0 \\ \gamma\frac{E}{B} & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \gamma & 0 & \gamma\frac{E}{B} & 0 \\ 0 & 1 & 0 & 0 \\ \gamma\frac{E}{B} & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -E & 0 & 0 \\ 0 & 0 & \gamma\frac{E^2}{B} - \gamma B & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\gamma = \frac{1}{\sqrt{1-\frac{E^2}{B^2}}} \Rightarrow \gamma\frac{E^2}{B} - \gamma B$$

$$= \gamma B\left(\frac{E^2}{B^2} - 1\right) = \frac{-\left(1-\frac{E^2}{B^2}\right)B}{\sqrt{1-\frac{E^2}{B^2}}}$$

$$= -\frac{B}{\gamma}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{B}{\gamma} & 0 \\ 0 & -\gamma\frac{E^2}{B}+\gamma B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-\gamma B\left(\frac{E^2}{B^2}-1\right) = \frac{B}{\gamma}$$

$$\Rightarrow F^{\alpha'\beta'} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{B}{\gamma} & 0 \\ 0 & \frac{B}{\gamma} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In this frame, there is no electric field, and only $B_z' = \frac{B}{\gamma}$.

$$\frac{dU^{\alpha'}}{d\tau} = \frac{q}{m} F^{\alpha'}{}_{\beta'} U^{\beta'}$$

$$F^{\alpha'}{}_{\beta'} = F^{\alpha'\gamma'} g_{\gamma'\beta'} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -B/\gamma & 0 \\ 0 & B/\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & B/\gamma & 0 \\ 0 & -B/\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{dU^{0'}}{d\tau} = 0 \qquad \frac{dU^{1'}}{d\tau} = \frac{q}{m} F^{1'}{}_{2'} U^{2'} = \frac{qB}{m\gamma} U^{2'} \qquad \frac{dU^{2'}}{d\tau} = \frac{q}{m} F^{2'}{}_{1'} U^{1'} = \frac{-qB}{m\gamma} U^{1'}$$

$$\frac{dU^{3'}}{d\tau} = 0 \qquad \frac{dU^{1'}}{d\tau} = \Omega U^{2'}, \qquad \frac{dU^{2'}}{d\tau} = -\Omega U^{1'}$$

$$\Omega = \frac{qB}{m\gamma}$$

Solution: $U^{1'} = A\sin(\Omega\tau + \phi)$

$U^{2'} = A\cos(\Omega\tau + \phi)$

$U^{0'} = \text{const.} = c_1$

$U^{3'} = \text{const} = c_2$

Boost back to lab frame:

$$\begin{bmatrix} u^0 \\ u^1 \\ u^2 \\ u^3 \end{bmatrix} = \begin{bmatrix} \gamma & 0 & -\gamma\frac{E}{B} & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma\frac{E}{B} & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ A\sin(\Omega\tau+\phi) \\ A\cos(\Omega\tau+\phi) \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} u^0 \\ u^1 \\ u^2 \\ u^3 \end{bmatrix} = \begin{bmatrix} \gamma c_1 - A\gamma \frac{E}{B}\cos(\Omega\tau + \phi) \\ A\sin(\Omega\tau + \phi) \\ -c_1\gamma\frac{E}{B} \quad + A\gamma\cos(\Omega\tau+\phi) \\ c_2 \end{bmatrix}$$

$$\frac{dx^\alpha}{d\tau} = u^\alpha \qquad \begin{aligned} x^0 &= \gamma c_1 \tau - \frac{A\gamma E}{\Omega B}\sin(\Omega\tau+\phi) + b_0 \\ x^1 &= -\frac{A}{\Omega}\cos(\Omega\tau+\phi) \quad + b_1 \\ x^2 &= -c_1\gamma\frac{E}{B}\tau \quad + \frac{A\gamma}{\Omega}\sin(\Omega\tau+\phi) + b_2 \\ x^3 &= c_2\tau \quad + b_3 \end{aligned}$$

At $\tau = 0$, $x^0 = 0,\ x^1 = 0,\ x^2 = 0,\ x^3 = 0$

$\Rightarrow \phi = 0, \quad b_0 = 0, \quad b_1 = \frac{A}{\Omega}, \quad b_2 = 0, \quad b_3 = 0$

At $\tau = 0$, $u^1 = 0,\ u^2 = 0,\ u^3 = 0, \quad u^0 = 1$ [Because $u \cdot u = u^\alpha u_\beta = 1$ always]

$u^2 = 0$: $-c_1\gamma\frac{E}{B} + A\gamma = 0 \qquad A = c_1\frac{E}{B}$

$u^0 = 1$: $\gamma c_1 - A\gamma\frac{E}{B} = \gamma\left(c_1 - c_1\frac{E^2}{B^2}\right) = c_1\gamma\left(1 - \frac{E^2}{B^2}\right) = \frac{c_1}{\gamma} \Rightarrow c_1 = \gamma$

$u^3 = 0$: $c_2 = 0$

$$\Rightarrow \boxed{\begin{aligned} u^0 &= \gamma^2 - \gamma^2\frac{E^2}{B^2}\cos(\Omega\tau) \\ u^1 &= \gamma\frac{E}{B}\sin(\Omega\tau) \\ u^2 &= -\gamma^2\frac{E}{B} + \gamma^2\frac{E}{B}\cos(\Omega\tau) \\ u^3 &= 0 \end{aligned}}$$

with $\gamma = \frac{1}{\sqrt{1 - E^2/B^2}}$

$u^\alpha = (\gamma_v, \gamma_v \vec{v})$

b. Solution found above

$$\boxed{\begin{aligned} x^0 &= \gamma^2\tau - \frac{\gamma^2}{\Omega}\frac{E^2}{B^2}\sin(\Omega\tau) \\ x^1 &= \frac{-\gamma}{\Omega}\frac{E}{B}\cos(\Omega\tau) \\ x^2 &= -\gamma^2\frac{E}{B}\tau + \frac{\gamma^2}{\Omega}\frac{E}{B}\sin(\Omega\tau) \\ x^3 &= 0 \end{aligned}}$$

$x^\alpha = (t, x, y, z)$