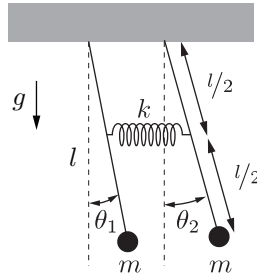


J09M.1 - Coupled Pendula

Problem

Two simple pendula, each of length l and mass m , are coupled by a spring of force constant k . The spring is attached to the rods of the pendula, which are massless and inextensible, at their half-way points, as shown. Throughout, assume the angles θ_1 and θ_2 are small and that motion is confined to the 2D plane.



- What are the normal frequencies of the system, and the corresponding normal mode vectors?
- Consider now the case of “weak coupling”—*i.e.*, the case when k is small. With respect to what is k small?

At $t = 0$ the lefthand pendulum is displaced by an angle $\theta_1(0) = \theta_0$ and released from rest; the righthand pendulum is at rest with $\theta_2(0) = 0$. Find expressions for $\theta_1(t)$ and $\theta_2(t)$ for $t > 0$. How long will it take before the lefthand mass stop swinging and the righthand mass achieves maximum amplitude?

J09M.2 - Minimizing Drag

Problem

The goal of this problem is to determine the optimum shape of a body in order to minimize the drag from a constant flow of air. Suppose that the body has cylindrical symmetry (that is, it is invariant under rotations around the z axis), and has a height L . If its radius is given by $r(z)$, a good approximation to the drag is the expression

$$D = a \int_0^L r(z) \left(\frac{dr}{dz} \right)^3 dz,$$

where $a > 0$ is some constant.

- a) If $r(0) = 0$ and $r(L) = d$, what is the optimal shape of the body in order to minimize D ?
- b) Suppose, in addition, that the body has a fixed volume V . How would you find the optimal shape under this constraint? Find a first order differential equation that the optimizing $r(z)$ will satisfy.

J09M.3 - Falling Rod

Problem

A massive uniform rod of length l starts to slide away from a frictionless vertical wall onto the frictionless horizontal floor. Assume that the bottom of the rod starts at rest in the corner, *i.e.*, the rod is initially vertical. Calculate the angle θ the rod makes with the wall when it first loses contact with a surface.

J09E.1 - Motion in EM Fields

Problem

In a large region of space there is a uniform magnetic field B in the z -direction and a uniform electric field E in the x -direction. A particle of mass m and charge q is initially at rest at the origin.

The equation of motion is

$$m \frac{dU^\alpha}{d\tau} = q F^\alpha{}_\beta U^\beta$$

where τ is the proper time of the particle and $U^\alpha = dx^\alpha/d\tau$ is its four-velocity. The field strength tensor is $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, where A^α is the 4-vector potential (its time component A^0 is the electric potential ϕ). Note that in this problem we use units where the speed of light $c = 1$.

- a) Solve for U^μ as a function of the proper time of the particle assuming that $B^2 > E^2$. What is the average 4-velocity of the particle?
- b) Solve for the particle position x^μ as a function of the proper time.

J09E.2 - Ion Source

Problem

Consider two infinite parallel plates separated by a distance a and with the gap between the plates filled with charged ions in vacuum. Assume that the motion of the ions is a one-dimensional laminar flow in the direction of the applied electric field. In the space-charge dominated limit, the electric field between the anode and the cathode is maximally shielded by the ion charge. Assume that the ions are initially emitted at the cathode $s = 0$ and travel to the anode at $s = a$ where they leave the plates through small negligible holes in the anode plate. The ion emission at the cathode maintains a static, steady state charge distribution between the plates and therefore a constant current at the anode plate. Let $V(s)$ be the electric potential at the position s between the plates.

- a) In the non-relativistic limit with laminar flow, write down the Poisson equation in terms of the current density J of the ions, the charge e and the mass m of the ion.
- b) For a space-charge dominated ion source, the condition of maximum space-charge shielding is equivalent to $V = 0$ and $dV/ds = 0$ at $s = 0$. What is the maximum current density J_{\max} for a given extraction voltage V_0 at the anode in the space-charge dominated ion source?
- c) What is the ratio of the maximum ion current density extracted for singly ionized gold atoms ($A = 79$) versus a proton ion source with the same extraction voltage V_0 in the space-charge dominated limit? Please estimate.

J09E.3 - Magnetic Pressure

Problem

Consider an infinite cylinder of conducting material with an axial current density $J_z = J(r)$ and a resulting azimuthal magnetic field $B_\phi = B(r)$. The coordinate r is the radial coordinate with respect to the axis of the cylinder. Assume the current is confined within the finite radius R of the cylinder. The integral of the current density over the cross-sectional area of the cylinder is the total current I .

- a) Use Ampère's law to determine $B_\phi = B(r)$ for $r \geq R$ in terms of I and r .
- b) Assume the current density is constant within the cylinder. Find $B(r)$ for $r < R$.
- c) The Lorentz forces on the current will "pinch" the material of the conductor and try to squeeze it radially inward. These pinching forces are balanced by non-magnetic pressure gradient forces (for example, elastic forces for a solid metal, or compressed-gas pressures for a plasma). Find an expression for the pressure, $p = p(r)$, inside the conductor for the uniform current distribution of part b), and sketch the dependence of p on r . You can assume that the pressure is zero at the surface of the conductor.
- d) Calculate and sketch the pressure for the case that the current flows in a very thin, uniform layer along the surface of the cylinder.

J09Q.1 - Dressed States and Rabi Oscillation

Problem

Rabi oscillation occurs in a quantum system when two states couple to an external field. Generally, the phenomenon can be understood from a semi-classical approach, which is quantum states + classical field. We are going to look at the problem by using a quantized field.

Considering a two-level spin system coupled to magnetic fields, the interaction Hamiltonian can be written as $H_1 = \gamma \vec{\sigma} \cdot \vec{B}$. Here γ is the coupling strength, $\vec{\sigma}$ represents Pauli matrices, and \vec{B} is the magnetic field. The spin states are $|\uparrow\rangle$ and $|\downarrow\rangle$. Assuming \vec{B} is composed of a DC longitudinal field $B - z\vec{z}$ and an AC transverse field $\vec{B}_T = B_x\vec{x} + B_y\vec{y}$, the interaction Hamiltonian is $H_1 = \gamma\sigma_z B_z + \gamma\vec{\sigma}_T \cdot \vec{B}_T$. To simplify the problem, we say that \vec{B}_T is a rotating field, and its classical form is $\vec{B}_T = B_r \cos \omega t \vec{x} + B_r \sin \omega t \vec{y}$ with frequency $\omega = 2\gamma B_z/\hbar$.

Now we want to use a quantized magnetic field, so \vec{B}_T is an operator, and $\vec{B}_T = B_r[(a + a^\dagger)\vec{x} + i(a - a^\dagger)\vec{y}]/2$, where a and a^\dagger are the ladder operators of photon state $|n\rangle$ of the magnetic field. Here, the photon energy is $\hbar\omega = 2\gamma B_z$. If there is no coupling between the transverse field and the spin, we find state $|\uparrow\rangle|n\rangle$ and state $|\downarrow\rangle|n+1\rangle$ are degenerate. The degeneracy is lifted into orthogonal dressed states when the coupling of the transverse field is present.

- Please use the full Hamiltonian (including the photon energy) to find the energies of the dressed states as functions of B_r .
- Assuming at $t = 0$ the probability of finding the system in the state $|\uparrow\rangle|n\rangle$ is 1 and in $|\downarrow\rangle|n+1\rangle$ is 0, how will the probabilities of the two states evolve (Rabi oscillation) for $t > 0$?

J09Q.2 - Scattering an Electron from a “Step” in the magnetic field

Problem

For $x < 0$, the magnetic field is $\vec{B} = 0$; for $x \geq 0$, $\vec{B} = B_0 \hat{z}$ is uniform and pointed along the z axis. An electron with its spin oriented along \hat{z} is incident from $x < 0$ with velocity $v \hat{x}$ and scatters from the field. Be sure to include both the interaction of the electron's charge and its magnetic moment with the field.

- a) What is the scattering wavefunction in the semi-classical WKB approximation in the classically-allowed region? Although you may leave the overall amplitude unnormalized, be sure to get the relative amplitude and phase of the incident and scattered waves correct. Clearly **state what gauge you use**, and **state what quantity the incident speed v must be much larger than** for the semi-classical WKB approximation to be appropriate in the region $x \geq 0$. Your expression for the wavefunction may involve an integral that can be performed, but carrying out the integral results in little simplification of the formula, so first give your result with the integral not performed and only carry out the integral if you have plenty of time.
- b) Explicitly show that for this wavefunction all components of the velocity of the electron at each position agree with the classical trajectory.

J09Q.3 - Trapping Atoms in an Optical Lattice

Problem

A 1-D optical lattice uses counter-propagating light beams to produce a standing wave, which has a periodic intensity pattern. Through the AC Stark-shift effect, the system energy is a function of the positions of the atoms. To simplify the problem, let's consider a hypothetical two-level atom with a ground state $|1\rangle$ and an excited state $|2\rangle$. The energy difference between the two states is $\hbar\omega_0$. Now an optical field $E(x, t) = E_0 \cos kx \cos \omega t$ is introduced, and most of the atoms are in state $|1\rangle$ right before the optical field is turned on. The frequency ω is close to ω_0 . The optical field interacts with atoms through the electric dipole coupling $\langle 2|E \cdot D|1\rangle = E \cdot \langle D \rangle$. Here, D is the dipole operator, and $|E \cdot \langle D \rangle| \ll |\hbar\omega_0 - \hbar\omega|$.

- a) Please find the perturbed energy of state $|1\rangle$ due to the optical field as a function of position. How deep is the potential well produced by the optical lattice?
- b) Where do the atoms tend to be trapped at a different optical frequency ω ?

J09T.1 - The Partition Function

Problem

A system A is in thermal equilibrium with a bath at temperature T . The thermal average of a physical quantity q is

$$\langle q \rangle = \sum_r q_r P_r,$$

where P_r , the probability that A occupies the state r of energy E_r , is given by

$$P_r = \frac{e^{-\beta E_r}}{Z},$$

where $\beta = 1/k_B T$, and the partition function $Z = \sum_r e^{-\beta E_r}$.

a) Show that the energy is given by

$$\langle E \rangle = -\frac{\partial \log Z}{\partial \beta}.$$

b) When the volume V of A is increased by dV at constant temperature, the energy of each state increases by $(\partial E_r / \partial V) dV$. The work done by A is $dW = -\langle \partial E_r / \partial V \rangle dV$. Show that the pressure p is given by

$$\langle p \rangle = \frac{1}{\beta} \frac{\partial \log Z}{\partial V}.$$

c) The free energy is defined as $F = E - TS$ with $S = -k_B \langle \log P \rangle$ the entropy. Derive an expression for F in terms of Z .

d) When observed over a long time, E fluctuates about $\bar{E} \equiv \langle E \rangle$. The average magnitude of the fluctuations is given by the variance $\langle (E - \bar{E})^2 \rangle$. Calculate the variance in terms of $\log Z$.

J09T.2 - Defects in a Lattice

Problem

In a lattice of N sites, each site is occupied by an atom at zero temperature. A lattice defect occurs when an atom moves to an interstitial site. The energy cost of a defect is Δ . At finite temperature T , we expect a finite number $\langle n(T) \rangle$ of defects to exist in equilibrium. Assume that defects do not interact with each other.

- a) Write down an expression for the partition function Z .
- b) Calculate $\langle n \rangle$ and the total free energy F of the lattice from Z at temperature T .
- c) Find the entropy $S(T)$ and heat capacity C_V from F .
- d) Use a purely statistical argument to rederive the entropy S starting with the total number of configurations W_n with n defects. Using your answer for $\langle n \rangle$, show that S agrees with part c).
- e) Use physical arguments to reproduce your answer for C_V in the low T limit ($\beta\Delta \gg 1$, where $\beta = 1/k_B T$).

J09T.3 - Errors in Gene Expression

Problem

A biased coin has a probability p of coming up heads in a single toss.

- Write down the probability P_n of obtaining n heads in N tosses.
- Show that, in the limit $p \ll 1$ and $n \ll N$, P_n reduces to the Poisson formula

$$P_n = \frac{\lambda^n}{n!} e^{-\lambda}, \quad (\lambda \equiv Np).$$

[Hint: Consider $\ln P_n$. Stirling's approximation is $n! \simeq n^n e^{-n}$. Note that $\lambda \gg p$.]

- Assume P_n has the Poisson form, and write $\sum_{n=0}^{\infty} n^s P_n = \langle n^s \rangle$. Calculate the mean $\bar{n} = \langle n \rangle$, and the variance $\langle (n - \bar{n})^2 \rangle$.
- A single strand of DNA is comprised of a string of basic units (the nucleotides A, T, G, and C) arrayed in a specific order. During gene expression, a molecular machine crawls along the strand and reads the units sequentially (this starts a chemical sequence that assembles one protein molecule).

The machine is highly reliable but not infallible. On average, it makes only one error for every 10^6 units read. Assume that a gene is comprised of $N_{\text{gene}} = 3 \times 10^4$ units. Calculate the probability that the machine makes zero errors when the gene is read. Find the probability that it makes 2 errors.