January 2008 Preliminary Exam, Mechanics Problem 3

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Problem:

When we derive Newton's equations of motion from a Lagrangian or Hamiltonian, the equations are invariant under time reversal, so that if $x(t)$ is a solution, so is $x(-t)$. If we add terms corresponding to damping or viscosity, the invariance is broken, and motions become obviously irreversible. Strangely, a form of reversibility is restored for fluid motion in the limit that viscosities are very large. Consider a fluid with viscosity η and density ρ, and assume that it is incompressible. The equations of motion are the Navier-Stokes equations,

$$\rho\left[\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}\right]=\nabla p+\eta\nabla^2\vec{v} \tag{1}$$

$$\nabla\cdot\vec{v}=0 \tag{2}$$

where $\vec{v}(\vec{x},t)$ is the velocity of the fluid element at position $\vec{x}$ at time t, and $p(\vec{x},t)$ is the pressure. To be concrete, imagine that we have a layer of fluid between two (large) parallel plates, a distance d apart.

(a) Let one of the plates move at velocity v_0, with the other plate held fixed. Now the natural unit of length is d, the natural unit of velocity is v_0, and the natural unit of pressure is ρv_0^2. Show that, in these natural units, a single term in the Navier-Stokes equations becomes dominant at large viscosity. Since viscosity has units, "large" means large relative to some characteristic scale η_C, which you should determine.

(b) In this limit of large viscosity (usually called the "low Reynolds number" limit, $Re\equiv\eta_C/\eta$, show that if the plate moves for a time T with velocity v_0, and then with velocity $-v_0$ for an equal time T, all elements of the fluid will be returned exactly to their initial locations, so that motion is reversible. You should show this explicitly for the problem of fluid between two plates (by solving the equations), and give a more general argument (which doesn't require solving the equations).

Notes: Recall that fluid immediately adjacent to a wall must move with the velocity of the wall - no slipping. The reversibility of motion means that if we inject a blob of dye into the fluid, then the motion of the wall at v_0 will spread the dye out (you should think about why) but then the motion at $-v_0$ will reassemble the original blob.

Solution:

(a) I need a characteristic time scale to estimate the first term, so let $t_c=d/v_0$. Then,

$$\rho\left[\frac{v_0}{t_c}+\left(\vec{v}_0\cdot\frac{\vec{v}_0}{d}\right)\right]=\frac{\rho v_0^2}{d}+\eta_c\frac{v_0}{d^2} \tag{3}$$

$$\implies \rho\left[\frac{v_0^2}{d}+\frac{v_0^2}{d}\right]=\frac{\rho v_0^2}{d}+\eta_c\frac{v_0}{d^2}\implies \eta_c=\rho d v_0 \tag{4}$$

(b) Let's neglect the times when the fluid is changing directions, or make some hand-wavy claim about symmetry or something. [1] Then, during each of the two periods of time, the fluid is in steady-state flow. The fluid is also incompressible, so all terms on the left side of the Navier-Stokes equation go to zero.

$$\implies -\nabla p=\eta\nabla^2\vec{v} \tag{5}$$

[1] Manson, Mark. *The Subtle Art of Not Giving a Fuck: A Counterintuitive Approach to Living a Good Life*. Harper One. 2016.

By symmetry, $\vec{v}$ can't have any in-plane variable dependence. Let the direction normal to the bottom plate be the $\hat{y}$ direction, and let $\vec{v}_0 = v_0\hat{x}$. Flow is laminar, so the pressure can only have $\hat{x}$ direction dependence (if this weren't true, planes of fluid would mix or you'd get inter-plane fluid turbulence). Therefore,

$$\frac{d^2v}{dy^2} = -\frac{1}{\eta}\frac{dp}{dx} \implies v(y) = -\frac{1}{2\eta}\frac{dp}{dx}y^2 + C_1 y + C_2 \tag{6}$$

First, in this problem, there's actually no pressure gradient in the $\hat{x}$ direction. Flow is caused by the top boundary's sheering motion, not by some force across an area in the y-z plane. Solving for the coefficients using the no-slip boundary conditions $\vec{v}(x, 0, z) = 0$ and $\vec{v}(x, d, z) = v_0\hat{x}$ yield the following:

$$\vec{v}(y) = \frac{v_0}{d}y\hat{x} \tag{7}$$

Therefore, if the top plane moves at $\vec{v} = v_0\hat{x}$ for time T and moves at $\vec{v} = -v_0\hat{x}$ for time T:

$$d(y) = v_0 yT + (-v_0)yT = 0 \tag{8}$$

Which implies the motion is reversible. If you don't believe it, definitely look it up on YouTube.

Qualitative argument: I could have probably reasoned this solution from thinking about how one x-y planar element of fluid interacts with the planar elements of fluid directly above and below it, which tells me the velocity distribution should be linear in y and pointing in the $\hat{x}$ direction. Boundary conditions can then be used to get the slope and intercept.