January 2008 Preliminary Exam, Electromagnetism Problem 1

Kevin P. Nuckolls (k.nuckolls@princeton.edu)

Problem:

An antenna consists of a circular wire loop of radius R, centered in the $x-y$ plane of a Cartesian coordinate system. The current has the same amplitude, $I = I(t)$, at all locations in the wire at a given time t. There is no net electrical charge on the wire. Assuming $\dot{I}$, the rate of change of the current, is slow enough that magnetic dipole radiation dominates any higher multipole, calculate:

(a) the vector potential $\vec{A} = \vec{A}(\vec{r}, t)$ and the scalar potential Φ at the location $\vec{r}$ and time t when $r \gg cI/\dot{I}$.

(b) the magnetic and electric field, $\vec{B}$ and $\vec{E}$, at $\vec{r}$ and t.

(c) the energy flux, $S = S(\theta, \phi)$, as a function of the polar angles θ and ϕ

(d) the total radiated power $P = \int S \sin(\theta) d\theta\, d\phi$.

Solution:

(a) According to the note at the bottom, I'll put everything in terms of the magnetic dipole moment, which, in SI units, is as follows:

$$\vec{m}(t) = IA\hat{z} = \pi R^2 I(t)\hat{z} \tag{1}$$

The vector potential of a magnetic dipole takes the following form:

$$\vec{A}(\vec{r}, t) = -\frac{\mu_0}{4\pi rc}\hat{r} \times \dot{\vec{m}}\left(t - \frac{r}{c}\right) \tag{2}$$

This is just a formula I have memorized. Along with the vector potentials for the electric dipole and the electric quadrupole, you can rederive anything they'd ask of you that could semi-reasonably be asked in 45 minutes, so yeah. Rewrite $\hat{z} = \cos(\theta)\hat{r} - \sin(\theta)\hat{\theta}$, so $\hat{r} \times \hat{z} = -\sin(\theta)\hat{\phi}$. Therefore,

$$\vec{A}(\vec{r}, t) = -\frac{\mu_0}{4\pi rc}\dot{m}\left(t - \frac{r}{c}\right)\sin(\theta)\hat{\phi} \tag{3}$$

There is no net charge on the loop (stated in problem), so

$$\Phi(\vec{r}, t) = 0 \tag{4}$$

(b) Determine the magnetic field directly from the vector potential:

$$\vec{B}(\vec{r}, t) = \nabla \times \vec{A} = \frac{\mu_0}{4\pi rc^2}\ddot{m}\left(t - \frac{r}{c}\right)\sin(\theta)\,(\hat{r} \times \hat{\phi}) = -\frac{\mu_0}{4\pi rc^2}\ddot{m}\left(t - \frac{r}{c}\right)\sin(\theta)\,\hat{\theta} \tag{5}$$

You can get the electric field from either the magnetic field or from the vector potential directly:

$$\vec{E}(\vec{r}, t) = -c\hat{r} \times \vec{B} = -\frac{\partial \vec{A}}{\partial t} = \frac{\mu_0}{4\pi rc}\ddot{m}\left(t - \frac{r}{c}\right)\sin(\theta)\hat{\phi} \tag{6}$$

(c) Use $\vec{E}$ and $\vec{B}$ to find the Poynting vector:

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} = \frac{c}{\mu_0}\left|\vec{B}\right|^2 \hat{r} = \frac{\mu_0}{16\pi^2 r^2 c^3}\left(\ddot{m}\left(t - \frac{r}{c}\right)\right)^2 \sin^2(\theta)\hat{r} \tag{7}$$

$$\implies S = \frac{\mu_0}{16\pi^2 c^3}\left(\ddot{m}\left(t - \frac{r}{c}\right)\right)^2 \sin^2(\theta) \tag{8}$$

(d) Oh, how nice of them to give us the formula for the power...

$$\begin{aligned} P &= \int S \sin(\theta)\, d\theta d\phi = \frac{\mu_0}{8\pi c^3}\left(\ddot{m}\left(t - \frac{r}{c}\right)\right)^2 \int_0^\pi \sin^3(\theta)\, d\theta \\ &= \frac{\mu_0}{8\pi c^3}\left(\ddot{m}\left(t - \frac{r}{c}\right)\right)^2 \int_{-1}^1 du\,(1 - u^2) \\ &= \frac{\mu_0}{6\pi c^3}\left(\ddot{m}\left(t - \frac{r}{c}\right)\right)^2 \end{aligned} \tag{9}$$

The denominator usually has a factor of 12, not 6 (which worried me for a second), but that's because we haven't time averaged $\ddot{m}$ yet, which isn't a factor of 1/2 for arbitrary magnetic moments.