Prelim solutions - January 2007

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1 Thermodynamics

1.1 Problem 3

Because probabilities are proportional to Boltzmann factors, at large temperatures higher energy states will have significant probabilities of being occupied. Since we only know information about low wavenumber (and hence low energies), we will have to assume low temperature for this problem.

(**a**) Here we want the average energy OF A SINGLE WAVE, that is, the energy of a quantum, weighted by the Bose-Einstein distribution:

$$< E >= \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1} \tag{1}$$

(**b**) To compute the heat capacity, we must first compute the internal energy. The formula for this is:

$$U = \int_0^{\omega_{max}} g(\omega)\hbar\omega\bar{n}_\omega \, d\omega \tag{2}$$

where ω_{max} is some maximum frequency, which can either be the Debye frequency or some upper limit due to the setup of the system. At low temperatures, only low-wavenumber states are significantly occupied, so we can put in, for the density of states, something that is valid only for low wavenumbers, since for higher wavenumbers the integrand will be close to 0 anyway. To get the density of states, use the following steps:

- 1 Consider an LxLxL box. Then we know that $k_x = 0, 2\pi/L, 4\pi/L$, etc. from basic quantum. The same is true for k_y and k_z. So each state has a "volume" in k-space of $(2\pi/L)^3 = 8\pi^3/V$ if V is the volume.

- 2 If the total number of states with energy less than E(k) is G(E), then $G(E)8\pi^3/V$ is the total k-space volume up to radius k.

- 3 If we have a dispersion relation k=f(E), then we can just plug in to get the number of states with energies less than or equal to E, namely:

$$G(E)8\pi^3/V = \frac{4\pi f(E)^3}{3}$$

- 4 Then the density of states (number of states with energy E), g(E), is just the derivative of G, namely:

$$g(E) = G'(E)$$

In our case, this gives a density of states:

$$g(\omega) = \frac{V}{(2\pi)^2 \hbar A}(\frac{\omega}{A})^{1/2}$$

The internal energy is, after changing variables from ω to x:

$$U = \frac{V(kT)^{5/2}}{(2\pi)^2 (\hbar A)^{3/2}} \int_0^{\hbar\omega_{max}/kT} \frac{x^{3/2}}{e^x - 1} dx \tag{3}$$

At low temperatures, the upper limit can be taken to be infinity. This integral has no analytic solution, so I will call it I. Since I is just a finite number, we can take a derivative with respect to temperature. Thus, the heat capacity will be proportional to $T^{3/2}$, which gives:

$$\alpha = 3/2 \tag{4}$$

(c) For insulators, this is the dominant term. For metals, on the other hand, the conduction electrons give another term which is linear in the temperature, and therefore dominant.