1 Thermodynamics

1.1 Problem 1

(a)

This is a chemical potential problem. We must know the chemical potential of an ideal gas, or at least know how to compute it relatively quickly. It is:

$$\mu = kTln\left(\frac{\rho\pi\hbar^3\sqrt{2mkT\pi}}{m^2(kT)^2}\right)$$

where ρ is the number density N/V. So in our case, taking into account the binding energy of hydrogen, we have:

$$-\epsilon + \mu_H(\epsilon = 0) = \mu_P + \mu_e$$

In this first part we are taking all the densities to be the same. If we assume $m_H = m_P$ (in reality $m_H = m_P * 1.0005$), we can cancel the hydrogen and proton terms:

$$kTln\left(\frac{\rho\pi\hbar^3\sqrt{2m_ekT\pi}}{m_e^2(kT)^2}\right) = -\epsilon$$

$$\rho = \frac{e^{-\epsilon/kT}m_e^2(kT)^2}{\pi\hbar^3\sqrt{2m_ekT\pi}} \approx 1.92\,10^{-17}m^{-3} \tag{1}$$

(b)

It's hard to interpret what they mean by "at this density". Do they mean "three times ρ"? Or the density of hydrogen atoms remains ρ while the others decrease? I don't know, but I'll take it to mean the latter, so that, after the reduction in temperature:

$$\rho_H = \rho$$

$$\rho_P = \rho_e = \frac{\rho}{9}$$

We plug these into the equation relating the chemical potentials:

$$-\epsilon + kTln\left(\frac{\rho\pi\hbar^3\sqrt{2m_HkT\pi}}{m_H^2(kT)^2}\right) = kTln\left(\frac{\rho\pi\hbar^3\sqrt{2m_PkT\pi}}{9m_P^2(kT)^2}\right) + kTln\left(\frac{\rho\pi\hbar^3\sqrt{2m_ekT\pi}}{9m_e^2(kT)^2}\right)$$

Applying the approximation $m_H \approx m_P$, we can cancel most of the proton and hydrogen terms. Then we plug in ρ from part (a), with T_i meaning the initial temperature:

$$-\epsilon = -kTln9 + kTln\left(\frac{e^{-\epsilon/kT_i}m_e^2(kT_i)^2\pi\hbar^3\sqrt{2m_ekT\pi}}{\pi\hbar^3\sqrt{2m_ekT_i\pi}9m_e^2(kT)^2}\right)$$

$$-\epsilon = -kTln9 + kTln\left(\frac{e^{-\epsilon/kT_i}T_i^{3/2}}{9T^{3/2}}\right)$$

$$kTln(e^{-\epsilon/kT}) = kTln\left(\frac{e^{-\epsilon/kT_i}T_i^{3/2}}{81T^{3/2}}\right)$$

$$e^{\epsilon/kT_i-\epsilon/kT} = \frac{T_i^{3/2}}{81T^{3/2}}$$

After staring at this equation for a long time, I realized the only way it can be satisfied is if T and T_i are close to each other. You can then write $T = T_i + \delta$, and solve for δ. You end up getting:

$$kT \approx 0.955kT_i = 0.130eV \quad (2)$$