Prelim solutions, a hand-waving approach - January 2007

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1 Quantum

1.1 Problem 1

Throughout this problem I use $\hbar = 1$, except when writing final results, where I restore the SI units.

(a) Finding the bound state of a delta function potential is a well known problem, and its solution can be found in [1].

$$E_b = \frac{m(V_0 a)^2}{2\hbar^2} \tag{1}$$

$$\Psi = \frac{(V_0 ma)^{1/2}}{\hbar} e^{-\sqrt{2mE_b}|x|} \tag{2}$$

(b) The eigenstates of a delta function potential in a well are of two types: even and odd. The even states are laborious to find (see M07Q3), but we'll see we only want the odd ones. The formula for the transmission coefficient, from time-dependent perturbation theory, is:

$$c_f = -i \int_{t_0}^{t} < \psi_f | H^1(t') | \psi_i > e^{i(E_f - E_i)t'} \, dt'$$

$$< \psi_f | H^1(t') | \psi_i > = \int_{-L/2}^{L/2} \psi_f(x) F x cos(\omega t') \frac{(V_0 ma)^{1/2}}{\hbar} e^{-\sqrt{2mE_b}|x|} \, dx$$

Since the potential is odd and the bound state is even, and since we are integrating from $-L/2$ to $L/2$, the integral will be 0 if the final state is even and (possibly) nonzero if it is odd. So we need continuum states that are odd. These are very simple: they are eigenstates which equal 0 at the origin, and therefore do not have discontinuous derivatives, i.e., they don't "see" the potential:

$$\psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{2n\pi x}{L}\right)$$

$$E_n = \frac{1}{2m}\left(\frac{2n\pi}{L}\right)^2$$

$$\langle\psi_f|H^1(t')|\psi_i\rangle = Fcos(\omega t')\sqrt{\frac{2}{L}}(V_0 ma)^{1/2}\int_{-L/2}^{L/2} sin\left(\frac{2n\pi x}{L}\right) xe^{-\sqrt{2mE_b}|x|}\,dx$$

$$\int_{-L/2}^{L/2} sin\left(\frac{2n\pi x}{L}\right) xe^{-\sqrt{2mE_b}|x|}\,dx = 2\int_{0}^{L/2} sin\left(\frac{2n\pi x}{L}\right) xe^{-\sqrt{2mE_b}x}\,dx =$$

$$= -i\int_0^{L/2}(e^{i2n\pi x/L} - e^{-i2n\pi x/L})xe^{-\sqrt{2mE_b}x}\,dx =$$

$$= -i\left[\left(\frac{e^{(ik-p)x}}{ik-p}(x-\frac{1}{ik-p})\right)_0^{L/2} + \left(\frac{e^{-(ik+p)x}}{ik+p}(x+\frac{1}{ik+p})\right)_0^{L/2}\right] =$$

$$= -i\left[\left(\frac{e^{(ik-p)L/2}}{ik-p}(L/2-\frac{1}{ik-p})\right) + \frac{1}{(ik-p)^2} + \left(\frac{e^{-(ik+p)L/2}}{ik+p}(L/2+\frac{1}{ik+p})\right) - \frac{1}{(ik+p)^2}\right]$$

where $k = 2n\pi/L$ and $p = \sqrt{2mE_b}$. Since $p > 0$, we notice that since in the end we will take $L \to \infty$, the exponential terms will vanish very fast. Therefore we can drop them now:

$$\int_{-L/2}^{L/2} sin(kx)xe^{-p|x|}\,dx = \frac{4kp}{(k^2+p^2)^2}$$

$$<\psi_f|H^1(t')|\psi_i>= Fcos(\omega t')\sqrt{\frac{2}{L}}(V_0 ma)^{1/2}\frac{4kp}{(k^2+p^2)^2}$$

$$c_f = -i\int_{t_0}^{t} Fcos(\omega t')\sqrt{\frac{2}{L}}(V_0 ma)^{1/2}\frac{4kp}{(k^2+p^2)^2}e^{i(E_f-E_i)t'}\,dt'$$

$$\int_{t_0}^{t} cos(\omega t')e^{i(E_f-E_i)t'}\,dt' = \frac{1}{2}\int_{t_0}^{t}(e^{i\omega t'}+e^{-i\omega t'})e^{i(E_f-E_i)t'}\,dt'$$

Now I invoke Horton's theorem of simplicity. If we try to leave the time finite, the answer to the integral is very complicated. Further, the problem says "find the transition probability", not "find the probability of finding it in the continuum at time t". So we assume that $t_0 \to -\infty$ and $t \to \infty$, and obtain:

$$\int_{t_0}^{t} cos(\omega t')e^{i(E_f-E_i)t'}\,dt' = \pi\left[\delta(\omega+E_f-E_i)+\delta(-\omega+E_f-E_i)\right]$$

$$c_f = -iF\sqrt{\frac{2}{L}}(V_0 ma)^{1/2}\frac{4kp}{(k^2+p^2)^2}\pi\left[\delta(\omega+E_f-E_i)+\delta(-\omega+E_f-E_i)\right]$$

$$\mathcal{P} = F^2\frac{2}{L}(V_0 ma)\frac{16k^2p^2}{(k^2+p^2)^4}\pi^2[\delta^2(\omega+E_f-E_i)+2\delta(\omega+E_f-E_i)\delta(-\omega+E_f-E_i)+\delta^2(-\omega+E_f-E_i)]$$

The middle term vanishes by the nature of delta functions. The other two terms are confuse: a squared delta function brings trouble... But invoking Tesileanu's theorem of simplicity, we interpret the square of the delta function to mean the same as the delta function:

$$\mathcal{P} = F^2 \frac{2}{L}(V_0 m a) \frac{16(2n\pi/L)^2 2mE_b}{((2n\pi/L)^2 + 2mE_b)^4} \pi^2 [\delta(\omega + \frac{1}{2m}(2n\pi/L)^2 - E_b) + \delta(-\omega + \frac{1}{2m}(2n\pi/L)^2 - E_b)]$$

This is the probability to go to a finite state of definite n, but we want the probability to go to *any* state, so we must sum over all the n's:

$$\mathcal{P} = \frac{F^2}{L} 4(V_0 a)^3 \pi^2 \left[\frac{E_b - \hbar\omega}{(2E_b - \hbar\omega)^4} + \frac{E_b + \hbar\omega}{(2E_b + \hbar\omega)^4} \right]$$

Note that we are left with an L in the denominator. That's too bad, I don't know what to do about it.

References

[1] D. Griffiths, *Introduction to Quantum Mechanics*.