# J07M.3 BUBBLE IN AN INCOMPRESSIBLE FLUID

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## 1. Problem

An explosion at time $t = 0$ in an ideal (zero viscosity) incompressible fluid produces a perfectly spherically symmetric expanding bubble of vacuum with radius $R(t)$ (neglect the effect of any gas or vapor inside the bubble). The bubble expands to maximum radius $R_{max}$ and then collapses. The pressure in the fluid far from the bubble is $P_\infty$, and the mass density of the fluid is $\rho$. Neglect any effects of surface tension or gravity; assume the bubble remains spherically symmetric at all times, and that the velocity field in the fluid is purely radial.

1.1. **(a.)** Obtain an expression for the velocity field inside the fluid, and obtain an expression for the total energy of the fluid in terms of $R$ and $\frac{dR}{dt}$.

Define $\vec{u}$ as the velocity field in the fluid. Since the fluid is incompressible, we have the property $\nabla \cdot \vec{u} = 0$ (if the divergence did not vanish, there would be a region with a net flow of fluid into or out of, which is not allowed). Because of this, we make an ansatz that $\vec{u}$ is the gradient of some function, say $\vec{u} = \nabla\phi$. Since we know $\nabla^2\phi = 0$, we want a purely radial 3D harmonic function. Perhaps from E&M we would try the harmonic function $\phi = \frac{f(t)}{r}$. Then $\vec{u} = -\frac{f(t)}{r^2}\hat{r}$.

Now, we have a boundary condition in that we know at the radius of the bubble $R$, the velocity is $\dot{R}$:

$$\dot{R} = -\frac{f(t)}{R^2} \to f(t) = -R^2\dot{R}$$

$$\vec{u} = \frac{R^2\dot{R}}{r^2}\hat{r}.$$

Kinetic energy is found by integrating over the entirety of the fluid the typical kinetic energy of each infinitesimal portion of fluid. This is performed from the radius of the bubble outwards, since the bubble itself is vacuum:

$$T = \int_R^\infty \frac{1}{2}\rho|\vec{u}|^2(4\pi r^2\, dr) = 2\pi\rho R^4\dot{R}^2\left(-\frac{1}{r}\bigg|_R^\infty\right) = 2\pi\rho R^3\dot{R}^2.$$

For potential energy, we must consider the implications of a bubble of vacuum in an incompressible fluid. The displaced fluid must go somewhere, which must cause displaced fluid at the edge of the fluid body very far away. Thus, the vacuum formed at the origin comes at the cost of an increase of potential equal to the volume of the bubble formed times the pressure at infinite radius (since the volume of the bubble must equal the volume of displaced fluid at the boundary). Thus, $U = \frac{4}{3}\pi R^3 P_\infty$:

$$E = T + U = 2\pi\rho R^3\dot{R}^2 + \frac{4}{3}\pi R^3 P_\infty$$

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1.2. **(b.)** Obtain an equation of motion for the bubble's radius $R(t)$.

The energy of the system is conserved. The bubble will reach a maximum radius when all the energy is potential, and thus at all radii the following is true:

$$2\pi\rho R^3\dot{R}^2 + \frac{4}{3}\pi R^3 P_\infty = \frac{4}{3}\pi R_{max}^3 P_\infty$$

which, when solved for $\dot{R}$, gives the equation of motion:

$$\frac{dR}{dt} = \left(P_\infty \frac{2}{3\rho}\left(\left(\frac{R_{max}}{R}\right)^3 - 1\right)\right)^{1/2}.$$

1.3. **(c.)** How long does it take for the bubble to collapse after it reaches its maximum radius?

Define $t_c$ as the time taken to collapse:

$$t_c = \int_{R_{max}}^0 \frac{dR}{\dot{R}} = \sqrt{\frac{3\rho}{2P_\infty}} \int_{R_{max}}^0 \left(\left(\frac{R_{max}}{R}\right)^3 - 1\right)^{-1/2} dR.$$

If we define the unitless value $x \equiv \frac{R_{max}}{R}$, straight substitution yields the relation:

$$t_c = R_{max}\sqrt{\frac{3\rho}{2P_\infty}} \int_\infty^1 \frac{-1/x^2}{\sqrt{x^3 - 1}}\, dx.$$

The problem allowed a finite dimensionless integral in the solution, which we have, although in this form the integral is not clearly finite. We can instead choose the unitless variable $y \equiv 1/x$, and get:

$$t_c = R_{max}\sqrt{\frac{3\rho}{2P_\infty}} \int_0^1 \frac{1}{\sqrt{(1/y)^3 - 1}}\, dy.$$

Either integral you choose, the solution to the integral is $\sqrt{\pi}\frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{3})}$, where $\Gamma$ is the Euler-Gamma function.

1.4. **(d.)** What is the asymptotic behavior of $R(t)$ in the final moments of the bubble's collapse when $R \ll R_{max}$? (Do not consider the possibility of "cold fusion"!)

When $R \ll R_{max}$, $\frac{R_{max}}{R} \gg 1$, so $\dot{R}$ goes approximately as:

$$\dot{R} \sim \frac{1}{R^{3/2}} \rightarrow \int R^{3/2}\, dR \sim t.$$

Thus, in the final moments of collapse, the bubble's radius goes asymptotically as:

$$R \sim t^{2/5}.$$