1 Mechanics

1.1 Problem 2

The kinetic energy term due to \( m \) is a little tricky. To get it correctly, we should write the coordinates of \( m \) (\( X \) and \( Y \)) in terms of \( x \) and \( \theta \). A little bit of trigonometry and looking at the picture tells you that:

\[
X = xc\cos\theta - L\sin\theta \\
Y = L\cos\theta - xs\sin\theta 
\]

\[
\dot{X} = \dot{x}\cos\theta - xs\sin\theta - L\cos\theta\dot{\theta} \\
\dot{Y} = -L\sin\theta\dot{\theta} + \dot{x}\sin\theta + xc\cos\theta\dot{\theta} 
\]

\[
\dot{X}^2 + \dot{Y}^2 = \dot{x}^2 + x^2\dot{\theta}^2 + L^2\dot{\theta}^2 - 2\dot{x}L\dot{\theta} 
\]

Since we want the linearized equations of motion, we want the lagrangian to contain terms of order up to 2. Thus,

\[
\dot{X}^2 + \dot{Y}^2 \approx \dot{x}^2 + L^2\dot{\theta}^2 - 2\dot{x}L\dot{\theta} 
\]

\[
\mathcal{L} = \frac{1}{2} ML^2\dot{\theta}^2 + \frac{1}{2} m(\dot{x}^2 + L^2\dot{\theta}^2 - 2\dot{x}L\dot{\theta}) + (M + m)gL\cos\theta + mgx\sin\theta 
\]

Applying the Euler-Lagrange equations gives the equations of motion:

\[
(M + m)L^2\ddot{\theta} - m\ddot{x}L = -(M + m)gL\dot{\theta} + mgx \quad (1) \\
-L\ddot{\theta} + \ddot{x} = g\dot{\theta} \quad (2) 
\]

These two equations can be written together as a matrix equation:

\[
M\ddot{\xi} = -K\xi 
\]

where \( \xi \) has components \( x \) and \( \theta \). We will guess that the solution is oscillatory, and so to get the eigenfrequencies, we have to solve the transcendental equation:

\[
\det(K - \omega^2M) = 0 
\]

I get two solutions from this equation \( \omega_1^2 = g/L \) and \( \omega_2^2 = -2mg/(ML) \). I was really careful in writing the matrices and the characteristic equation, so I don’t know how I reached this problem. It could be, also, that my solution is correct, the negative \( \omega^2 \) meaning that the corresponding eigenmode is of exponential form, but I don’t think so. I’ll wait for opinions.