Prelim solutions - January 2007

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1 Electromagnetism

1.1 Problem 3

1.2 (a)

Electromagnetic wave, frequency ω, linear medium. Take a guess $\boldsymbol{E} = \boldsymbol{E_0} exp[ikx - \omega t]$. Need a wave equation:

$$\nabla \times (\nabla \times E) = -\frac{\partial}{\partial t}(\nabla \times B)$$
$$\nabla(\nabla \cdot E) - \nabla^2 E = -\frac{\partial}{\partial t}\left(\mu\sigma E + \mu\epsilon\frac{\partial E}{\partial t}\right)$$

The latter comes from $J_f = \sigma E$, where σ is the conductivity.

$$\nabla^2 \boldsymbol{E} = \mu(\sigma\dot{\boldsymbol{E}} + \epsilon\ddot{\boldsymbol{E}})$$

Usually would say $J_f = 0$, but we don't know $\sigma = 0$ (at least for now). With the guess taken above:

$$\nabla^2 E = -k^2 E$$
$$\dot{E} = -i\omega E$$
$$\ddot{E} = -\omega^2 E$$
$$k^2 = \mu(i\sigma\omega + \epsilon\omega^2)$$
$$k^2 = \mu\omega(\omega\epsilon + i\sigma)$$

If k has an imaginary component, $E \to 0$ as the wave moves (from the guess we took). Thus, we need $\sigma = 0$ because we are told there is no loss.

$$k = |\omega|\sqrt{\mu\epsilon}$$

We made a guess, so let's say $|\omega| = \omega$.

$$k = \omega\sqrt{\mu\epsilon} \tag{1}$$

Since we are told $\mu\epsilon > 0$, k is real as predicted. We can easily check that our solution is divergence-less, as required, as long as $E_{0_x} = 0$. That means, as expected, that the electric field is perpendicular to the direction of motion. Let y be the direction of the electric field, so that:

$$\boldsymbol{E} = E_0 cos(\omega\sqrt{\mu\epsilon}x - \omega t)\hat{y} \tag{2}$$

By Faraday's law, we can find the magnetic field to be:

$$\boldsymbol{B} = \sqrt{\mu\epsilon}E_0[cos(\omega\sqrt{\mu\epsilon}x - \omega t) + C]\hat{z} \tag{3}$$

How do you determine this constant C?

And then from the definitions of $\boldsymbol{D}$ and $\boldsymbol{H}$, we can get:

$$\boldsymbol{D} = E_0 \epsilon cos(\omega\sqrt{\mu\epsilon}x - \omega t)\hat{y} \tag{4}$$

$$\boldsymbol{H} = \sqrt{\frac{\epsilon}{\mu}}E_0[cos(\omega\sqrt{\mu\epsilon}x - \omega t) + C]\hat{z} \tag{5}$$

(b)

$$v_g = \frac{d\omega}{dk} = \frac{1}{dk/d\omega} = 2\sqrt{\omega^2\epsilon\mu}\frac{1}{\frac{d(\omega\epsilon)}{d\omega}\omega\mu + \omega\epsilon\frac{d(\omega\mu)}{d\omega}} < 0 \tag{6}$$

$$\frac{(\omega/k)}{v_g} = \frac{1}{k}\frac{1}{2\sqrt{\epsilon\mu}}\left(\frac{d(\omega\epsilon)}{d\omega}\omega\mu + \omega\epsilon\frac{d(\omega\mu)}{d\omega}\right) = \frac{1}{2}\left(\frac{1}{\epsilon}\frac{d(\omega\epsilon)}{d\omega} + \frac{1}{\mu}\frac{d(\omega\mu)}{d\omega}\right) \tag{7}$$

(c)

Laser, narrow, ω. $v_g, \epsilon, \mu < 0$. The derivation is the same as for the usual case (positive constants). The boundary conditions must hold at all points of the boundary, and so, at $x = 0$, the argument in the cosines must be equal:

$$\boldsymbol{k}_i \cdot \boldsymbol{r} = \boldsymbol{k}_r \cdot \boldsymbol{r} \quad x = 0$$

This means that the components of $\boldsymbol{k}$ parallel to the surface must be equal. Set $z = 0$, then:

$$\frac{1}{c}sin\theta_i = \sqrt{\mu\epsilon}sin\theta_r$$

$$\theta_r = arcsin\left(\frac{1}{c\sqrt{\mu\epsilon}}sin\theta_i\right) \tag{8}$$