J07E.2 - Rotating Shell of Charge (Solution by Jim Wu)

A hollow spherical shell centered at the origin has radius a and a total electric charge $Q > 0$ uniformly distributed over its surface. The shell is slowly spun up to an angular velocity $\omega = \omega\hat{z}$ (where $\omega_0 > 0$) over a period of time $\tau \gg a/c$, where c is the speed of light, so radiation effects can be ignored.

(a) To linear order in $d\omega/dt$, find the expressions for the electromagnetic fields $\vec{E}(\vec{r})$ and $\vec{B}(\vec{r})$ throughout space, as functions of ω and $d\omega/dt$. Make a qualitatively correct sketch showing the pattern of the electric field lines in the plane $z = 0$. Indicate the direction of rotation of the charged shell on your plot.

(b) After the angular velocity ω_0 is reached, what is the total angular momentum $\vec{L}$ stored in the electromagnetic fields?

Solution:

(a) In the quasistatic case (without any consideration for radiation), the Coulombic electric field is given by Gauss's law. For $r \geq a$, the electric field is

$$E_r(4\pi r^2) = \frac{Q}{\epsilon_0} \qquad \Rightarrow \qquad \vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$$

and for $r < R$, since the sphere is hollow and has no charge inside, then $E = 0$. So,

$$\vec{E} = \begin{cases} 0 & \text{for } r < a \\ \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} & \text{for } r \geq a \end{cases}$$

However, this is just the Coulomb field which does not take into account the Faraday effect from the time changing magnetic field. So, let's start by finding the magnetic field for rotating charged sphere.

Since the currents are confined to the surface of the spherical shell, then we have $\nabla \times \vec{B} = 0$ everywhere else. Hence, we can propose that $\vec{B} = -\nabla\phi$ for some magnetic scalar potential ϕ. Since magnetic field vanishes at infinity, then we require ϕ go to infinite as $r \to \infty$. Furthermore, we also require that ϕ be finite at $r = 0$. Since our current configuration has azimuthal symmetry, then the potential must be of the form

$$\phi(r < a) = \sum_\ell A_\ell r^\ell P_\ell(\cos\theta)$$
$$\phi(r > a) = \sum_\ell \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

Gauss's law for magnetism $\nabla \cdot \vec{B}$ suggests that the components of $\vec{B}$ normal to the surface are continuous. Hence, there is continuity of $\frac{\partial\phi}{\partial r}$ at $r = a$, which means that that

$$\begin{aligned}
\frac{\partial\phi(r=a_+)}{\partial r} &= \frac{\partial\phi(r=a_-)}{\partial r} \\
\sum_\ell \ell A_\ell a^{\ell-1} P_\ell(\cos\theta) &= \sum_\ell -(\ell+1)\frac{B_\ell}{a^{\ell+2}} P_\ell(\cos\theta) \\
B_\ell &= -\frac{\ell}{\ell+1} a^{2\ell+1}
\end{aligned}$$

Furthermore, since there is a free current density at the surface of the spherical shell, then from $\nabla \times \vec{B} = \mu_0 \vec{J}$, there is a discontinuity in the θ derivatives of the potential:

$$B(r=a_+) - B(r=a_-) = -\frac{1}{a}\frac{\partial\phi(r=a_+)}{\partial\theta} + \frac{1}{a}\frac{\partial\phi(r=a_-)}{\partial\theta} = \mu_0 K_\phi$$

where the surface current density is $K_\phi = \sigma\omega a \sin\theta$. Since K_ϕ is dependent on $\sin\theta$, then this suggests that only the $\ell = 1$ terms in the magnetic scalar potential remains. So,

$$\begin{aligned}
-\frac{1}{a}\left(\frac{A_1}{2} a\sin\theta\right) + \frac{1}{a}\left(-A_1 a\sin\theta\right) &= \mu_0\sigma\omega a\sin\theta \\
A_1 &= -\frac{2}{3}\mu_0\sigma\omega a \\
B_1 &= \frac{1}{3}\mu_0\sigma\omega a^4
\end{aligned}$$

The magnetic scalar potential is

$$\phi(r,\theta) = \begin{cases} -\frac{2\mu_0\sigma\omega a}{3} r\cos\theta & \text{for } r < a \\ \frac{\mu_0\sigma\omega a^4}{3}\frac{\cos\theta}{r^2} & \text{for } r > a \end{cases}$$

and the magnetic field is

$$\vec{B} = -\nabla\phi = -\frac{\partial\phi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\phi} = \begin{cases} \frac{2\mu_0\sigma\omega a}{3}(\cos\theta\hat{r} - \sin\theta\hat{\theta}) & \text{for } r < a \\ \frac{\mu_0\sigma\omega a^4}{3}\left(\frac{2\cos\theta}{r^3}\hat{r} + \frac{\sin\theta}{r^3}\hat{\theta}\right) & \text{for } r > a \end{cases}$$

Applying the fact that $\sigma = Q/4\pi a^2$, we can rewrite the magnetic field as

$$\vec{B} = \begin{cases} \frac{\mu_0 Q\omega}{6\pi a}(\cos\theta\hat{r} - \sin\theta\hat{\theta}) & \text{for } r < a \\ \frac{\mu_0 Q\omega a^2}{12\pi r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}) & \text{for } r > a \end{cases}$$

Notice that $\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}$, which tells us that the magnetic field is constant and points in the $\hat{z}$ direction inside. Furthermore, the magnetic field looks like a magnetic dipole outside, as expected.

Now we must use this $\vec{B}$ to determine the Faraday correction to the electric field $\vec{E}_f$. Note that

$$\nabla \times \vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \vec{A}) = -\nabla \times \frac{\partial\vec{A}}{\partial t}$$

Furthermore, since there are no charges inside our outside the spherical shell, $\nabla \cdot E = 0$ and we can also choose for $\nabla \cdot \vec{A} = 0$, implying that $\nabla \cdot \vec{E} = -\nabla \cdot \frac{\partial \vec{A}}{\partial t}$. Since the divergence and curl of a vector are equal to each other, then we can say that

$$\vec{E}_f = -\frac{\partial \vec{A}}{\partial t}$$

We have two equivalent formulations of $\vec{B}$, that is

$$-\nabla\phi = \nabla \times \vec{A}$$

We know that $\vec{A}$ is in $\hat{\phi}$ since the magnetic vector potential points in the direction of the current density. So, we have two equations:

$$\begin{aligned} -\frac{\partial \phi}{\partial r} &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\phi \sin\theta) \\ -\frac{1}{r}\frac{\partial\phi}{\partial\theta} &= -\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) \end{aligned}$$

Let's work with the second equation. Inside the sphere,

$$\begin{aligned} \frac{\partial}{\partial r}(rA_\phi) &= \frac{\mu_0 Q\omega r\sin\theta}{6\pi a} \\ rA_\phi &= \frac{\mu_0 Q\omega r^2 \sin\theta}{12\pi a} \\ A_\phi &= \frac{\mu_0 Q\omega r\sin\theta}{12\pi a} \end{aligned}$$

and outside the sphere,

$$\begin{aligned} \frac{\partial}{\partial r}(rA_\phi) &= -\frac{\mu_0 Q\omega a^2 \sin\theta}{12\pi r^2} \\ rA_\phi &= \frac{\mu_0 Q\omega a^2\sin\theta}{12\pi r} \\ A_\phi &= \frac{\mu_0 Q\omega a^2 \sin\theta}{12\pi r^2} \end{aligned}$$

Hence the Faraday correction to the electric field is

$$\vec{E}_f = -\frac{\partial \vec{A}}{dt} = \begin{cases} -\frac{\mu_0 Q r\sin\theta}{12\pi a}\frac{d\omega}{dt}\hat{\phi} & \text{for } r < a \\ -\frac{\mu_0 Q a^2 \sin\theta}{12\pi r^2}\frac{d\omega}{dt}\hat{\phi} & \text{for } r > a \end{cases}$$

Of course, we can take this and obtain a correction for the magnetic field, but it would yield a $\frac{d^2\omega}{dt^2}$ term, which is a second order term. Keeping up to linear orders of $d\omega/dt$, the fields are

$$\vec{E} = \begin{cases} -\frac{\mu_0 Q r\sin\theta}{12\pi a}\frac{d\omega}{dt}\hat{\phi} & \text{for } r < a \\ \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} - \frac{\mu_0 Q a^2\sin\theta}{12\pi r^2}\frac{d\omega}{dt}\hat{\phi} & \text{for } r > a \end{cases}$$

$$\vec{B} = \begin{cases} \frac{\mu_0 Q\omega}{6\pi a}(\cos\theta\hat{r} - \sin\theta\hat{\theta}) & \text{for } r < a \\ \frac{\mu_0 Q\omega a^2}{12\pi r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}) & \text{for } r > a \end{cases}$$

(b) Once the spherical shell has stopped rotating, then the fields no longer depend on $d\omega/dt$ (as it is zero) and are given by

$$\vec{E} = \begin{cases} 0 & \text{for } r < a \\ \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} & \text{for } r > a \end{cases}$$

$$\vec{B} = \begin{cases} \frac{\mu_0 Q\omega_0}{6\pi a}(\cos\theta\hat{r} - \sin\theta\hat{\theta}) & \text{for } r < a \\ \frac{\mu_0 Q\omega_0 a^2}{12\pi r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}) & \text{for } r > a \end{cases}$$

So, the linear momentum density stored in the fields is

$$\vec{\pi} = \epsilon_0 \vec{E} \times \vec{B} = \begin{cases} 0 & \text{for } r < a \\ \frac{\mu_0 Q^2 \omega_0 a^2}{48\pi^2 r^5}\sin\theta\hat{\phi} & \text{for } r > a \end{cases}$$

and the angular momentum density stored in the fields is

$$\vec{l} = \vec{r} \times \vec{\pi} = r\hat{r} \times \vec{\pi} = \begin{cases} 0 & \text{for } r < a \\ -\frac{\mu_0 Q^2 \omega_0 a^2}{48\pi^2 r^4}\sin\theta\hat{\theta} & \text{for } r > a \end{cases}$$

Hence, the total angular momentum stored in the fields is

$$\begin{aligned}
\vec{L} &= \int \vec{l}\, d^3r \\
&= -\frac{\mu_0 Q^2 \omega_0 a^2}{48\pi^2} \int_a^\infty \int_0^\pi \frac{\sin\theta}{r^4}(2\pi r^2 \sin\theta)(-\sin\theta\hat{z})\, dr d\theta \\
&= \frac{\mu_0 Q^2 \omega_0 a^2}{24\pi} \int_a^\infty \int_0^\pi \frac{\sin^3\theta}{r^2}\, dr\, d\theta\, \hat{z} \\
&= \frac{\mu_0 Q^2 \omega_0 a}{24\pi} \int_0^\pi \sin\theta - \sin\theta\cos^2\theta\, d\theta\, \hat{z} \\
&= \frac{\mu_0 Q^2 \omega_0 a}{24\pi} \left[-\cos\theta + \frac{1}{3}\cos^3\theta \right]_0^\pi \hat{z} \\
&= \frac{\mu_0 Q^2 \omega_0 a}{18\pi}\, \hat{z}
\end{aligned}$$

Note that all the $\hat{x}$ and $\hat{y}$ components cancel out by symmetry, leaving us with only the z component.

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