1 Electromagnetism

1.1 Problem 1

This is a problem relating to the method of images. Let's begin by obtaining the potential due to the two charges everywhere, for both $r > a$ and $r < a$.

- $r > a$: This is the case most commonly discussed in the textbooks [?]. The potential is found for the case when the sphere is grounded, and then a point charge is added at the center to account for the total charge on the sphere.

  To make the potential be 0 on the sphere, put an image charge q at a position d from the center, along the line connecting the center of the sphere with the point charge Q. Of course, $d < a$ in order to make the image charge plausible. The potential due to the two charges is, at a distance r' from the center, at an angle θ from the line joining the center and the point charge Q, is:

$$V(\boldsymbol{r'}) = \frac{Q}{4\pi\epsilon_0\sqrt{r^2 + r'^2 - 2r'rcos\theta}} + \frac{q}{4\pi\epsilon_0\sqrt{r'^2 + d^2 - 2r'dcos\theta}} \tag{1}$$

  We find that the choices:

$$d = \frac{a^2}{r}$$
$$q = -\frac{Qa}{r}$$

  the potential becomes:

$$V(\boldsymbol{r'}) = \frac{Q}{4\pi\epsilon_0\sqrt{r^2 + r'^2 - 2r'rcos\theta}} - \frac{Qa/r}{4\pi\epsilon_0\sqrt{r'^2 + \frac{a^4}{r^2} - 2\frac{r'a^2}{r}cos\theta}} \tag{2}$$

  which is 0 at $r' = a$. This would be the total potential if the sphere were grounded. Since it is not, and since the total charge on it is Q, we need to have a total of Q inside the spherical surface of radius a around the origin. We currently have q inside it, so if we put a charge of value $Q - q$ at the center, we accomplish this. The total potential then becomes:

$$V(\boldsymbol{r'}) = \frac{Q(1 + \frac{a}{r})}{4\pi\epsilon_0 r'} + \frac{Q}{4\pi\epsilon_0\sqrt{r^2 + r'^2 - 2r'rcos\theta}} - \frac{Qa/r}{4\pi\epsilon_0\sqrt{r'^2 + \frac{a^4}{r^2} - 2\frac{r'a^2}{r}cos\theta}} \tag{3}$$

  It is a good exercise to check that the total charge on the sphere is indeed Q. For that, we need to calculate the first partial derivatives of V:

$$\frac{\partial V}{\partial r'} = -\frac{Q(1 + \frac{a}{r})}{4\pi\epsilon_0 r'^2} - \frac{Q(2r' - 2rcos\theta)}{4\pi\epsilon_0(r'^2 + r^2 - 2r'rcos\theta)^{3/2}} + \frac{Qa/r(r' - \frac{a^2}{r}cos\theta)}{4\pi\epsilon_0(r'^2 + \frac{a^4}{r^2} - 2\frac{r'a^2}{r}cos\theta)^{3/2}} \tag{4}$$

$$\frac{\partial V}{\partial \theta} = -\frac{Q(r' r sin\theta)}{4\pi\epsilon_0(r'^2 + r^2 - 2r' r cos\theta)^{3/2}} + \frac{Qa/r(\frac{r' a^2}{r} sin\theta)}{4\pi\epsilon_0(r'^2 + \frac{a^4}{r^2} - 2\frac{r' a^2}{r} cos\theta)^{3/2}} \tag{5}$$

$$\frac{\partial V}{\partial \theta}(r = a) = -\frac{Q(arsin\theta)}{4\pi\epsilon_0(a^2 + r^2 - 2arcos\theta)^{3/2}} + \frac{Qarsin\theta}{4\pi\epsilon_0(r^2 + a^2 - 2arcos\theta)^{3/2}} = 0 \tag{6}$$

The electric field is:

$$\boldsymbol{E} = -\nabla V \tag{7}$$

Thus, the last of the equations above confirms that the electric field points radially outward from the sphere, as expected since it is a conductor. Thus:

$$\boldsymbol{E} = E\hat{r} \tag{8}$$

$$\boldsymbol{E}(a) = \frac{Q(1 + \frac{a}{r})}{4\pi\epsilon_0 a^2} - \frac{Q(a - rcos\theta)}{4\pi\epsilon_0(a^2 + r^2 - 2arcos\theta)^{3/2}} + \frac{Qa(a - a^2 cos\theta/r)/r}{4\pi\epsilon_0(a^2 + \frac{a^4}{r^2} - \frac{2a^3 cos\theta}{r})^{3/2}} \tag{9}$$

We know that the potential inside a closed conductor is constant if there are no free charges in the vacuum enclosed by the conductor. Thus, the electric field inside our physical conducting sphere will be zero. This says $\boldsymbol{E}_{in} = \boldsymbol{0}$, and therefore the surface charge density is:

$$\begin{aligned}
\sigma = \frac{Q(1 + \frac{a}{r})}{4\pi a^2} + \frac{Q(a - rcos\theta)}{4\pi(a^2 + r^2 - 2arcos\theta)^{3/2}} - \frac{Q(\frac{r}{r^2} - \frac{acos\theta}{r^2})}{4\pi a(1 + \frac{a^2}{r^2} - \frac{2acos\theta}{r})^{3/2}} \\
= \frac{Q(1 + \frac{a}{r})}{4\pi a^2} + \frac{Q(a - rcos\theta)}{4\pi(a^2 + r^2 - 2arcos\theta)^{3/2}} - \frac{Q(r - acos\theta)r}{4\pi a(r^2 + a^2 - 2arcos\theta)^{3/2}} \\
= \frac{Q(1 + \frac{a}{r})}{4\pi a^2} + \frac{Q}{4\pi(a^2 + r^2 - 2arcos\theta)^{3/2}}(a - \frac{r^2}{a}) \\
= \frac{Q(1 + \frac{a}{r})}{4\pi a^2} + \frac{Q(a^2 - r^2)}{4\pi a(a^2 + r^2 - 2arcos\theta)^{3/2}}
\end{aligned} \tag{10}$$

Integrate this to get the total charge on the sphere:

$$\int_0^\pi \frac{sin\theta}{(a^2 + r^2 - 2arcos\theta)^{3/2}} d\theta$$
$$\mu := a^2 + r^2 - 2arcos\theta$$
$$d\mu = 2arsin\theta d\theta$$

The integral becomes:

$$\int_{(a-r)^2}^{(a+r)^2} \frac{1/2ar}{\mu^{3/2}}\, d\mu = \frac{1}{ar}[\frac{((a+r)^2)^{-1/2}}{-1} - \frac{((a-r)^2)^{-1/2}}{-1}] = \frac{-1}{ar}(\frac{1}{a+r} - \frac{1}{r-a})$$
$$= \frac{-1}{ar}(\frac{r-a-r-a}{r^2-a^2}) = \frac{2}{r(r^2-a^2)}$$

$$Q_T = \int_0^{2\pi}\int_0^{\pi} \sigma a^2 sin\theta\, d\theta d\phi$$
$$= Q(1+a/r) + \frac{Q(a^2-r^2)2\pi a^2 2}{4\pi ar(r^2-a^2)} = Q(1+\frac{a}{r}) - \frac{Qa}{r} = Q \tag{11}$$

This proves that you must put a point charge at the center to get the charge on the sphere to be Q. We have the potential outside the sphere for the case when the test charge is outside the sphere. We also know that the potential inside is just a constant. The continuity of V will say what this constant should be:

$$V_{out}(a) = \frac{Q(1+a/r)}{4\pi\epsilon_0 a}$$
$$\rightarrow V_{in}(\boldsymbol{r'}) = V_{in}(a) = \frac{Q(1+a/r)}{4\pi\epsilon_0 a}$$

Note that the potential outside the sphere vanishes for $r' \rightarrow \infty$. Finally, we can do a reality check by noting that the potential for $r \rightarrow \infty$ should be the the potential due to a sphere of uniformly distributed charge Q:

$$V_{out}(r=\infty) = \frac{Q}{4\pi\epsilon_0 r'} \tag{12}$$
$$V_{in}(r=\infty) = \frac{Q}{4\pi\epsilon_0 a} \tag{13}$$

which are exactly as expected.

- *$r < a$:*

  This case is solved for a grounded sphere in Wikipedia [**?**]. The expressions for the charge and position of the image charge are the same as for the previous case, except since now $a > r$, the image charge will be outside the sphere, as should be. In fact, the expression for the potential that vanishes at the sphere is exactly the same as above (2). Now, in order to have the total charge on the sphere be Q, we must add a uniformly distributed image charge at the location of the conducting sphere, with total charge $Q - q$. The potential inside a uniformly charged sphere is just a constant. The total potential is:

$$V(\boldsymbol{r'}) = \frac{Q(1+\frac{a}{r})}{4\pi\epsilon_0 a} + \frac{Q}{4\pi\epsilon_0\sqrt{r^2+r'^2-2r'rcos\theta}} - \frac{Qa/r}{4\pi\epsilon_0\sqrt{r'^2+\frac{a^4}{r^2}-2\frac{r'a^2}{r}cos\theta}} \tag{14}$$

To check the total charge on the surface:

$$E_{in}(\boldsymbol{r'}) = \frac{Q(r' - rcos\theta)}{4\pi\epsilon_0(r^2 + r'^2 - 2rr'cos\theta)^{3/2}} - \frac{Q(\frac{r'r^2}{a^2} - rcos\theta)}{4\pi\epsilon_0(a^2 + \frac{r'^2r^2}{a^2} - 2rr'cos\theta)^{3/2}}$$

$$E_{in}(a) = \frac{Q(a - rcos\theta)}{4\pi\epsilon_0(r^2 + a^2 - 2racos\theta)^{3/2}} - \frac{Q(r^2 - racos\theta)/a}{4\pi\epsilon_0(a^2 + r^2 - 2racos\theta)^{3/2}}$$

$$= \frac{Q(a^2 - r^2)}{4\pi\epsilon_0 a(r^2 + a^2 - 2racos\theta)^{3/2}} \tag{15}$$

$$E_{out} = \frac{2Q}{4\pi\epsilon_0 r'^2}$$

$$E_{out}(a) = \frac{2Q}{4\pi\epsilon_0 a^2} \tag{16}$$

$$\sigma = \frac{2Q}{4\pi a^2} - \frac{Q(a^2 - r^2)}{4\pi a(r^2 + a^2 - 2racos\theta)^{3/2}} \tag{17}$$

$$Q_T = 2Q + \frac{(a^2 - r^2)Q}{2r}[\frac{1}{sqrt(a + r)^2} - \frac{1}{sqrt(a - r)^2}] = 2Q + \frac{(a^2 - r^2)}{2r}[\frac{1}{a + r} - \frac{1}{a - r}]Q$$

$$= 2Q + \frac{(-r - r)Q}{2r} = 2Q - Q = Q \tag{18}$$

We now have the potential inside the sphere, but we have to get the potential outside as well. We know that for a conducting with total charge Q on the surface and total charge Q inside, the potential outside the conductor is:

$$V_{out}(\boldsymbol{r'}) = \frac{2Q}{4\pi\epsilon_0 r'} \tag{19}$$

We must check the continuity of V and then make the reality check for $r = 0$.

$$V_{out}(a) = \frac{2Q}{4\pi\epsilon_0 a} \tag{20}$$

$$V_{in}(a) = \frac{Q(1 + a/r)}{4\pi\epsilon_0 a} \tag{21}$$

We have run into a bit of trouble. But let's ignore it for a moment and check the case $r = 0$:

$$V_{in}(r = 0) = \frac{Q(1 + \frac{a}{r})}{4\pi\epsilon_0 a} + \frac{Q}{4\pi\epsilon_0 r'} - \frac{Q}{4\pi\epsilon_0 a} = \frac{Q}{4\pi\epsilon_0 r} + \frac{Q}{4\pi\epsilon_0 r'} \tag{22}$$

which is infinite! This is definitely unrealistic. Notice that we can fix both problems at the same time (and get the expected answer for the potential in the case $r = 0$ if we change the $1 + a/r$ to 2 in the first term of the potential:

$$V_{in}(\boldsymbol{r'}) = \frac{2Q}{4\pi\epsilon_0 a} + \frac{Q}{4\pi\epsilon_0\sqrt{r^2 + r'^2 - 2r'rcos\theta}} - \frac{Qa/r}{4\pi\epsilon_0\sqrt{r'^2 + \frac{a^4}{r^2} - 2\frac{r'a^2}{r}cos\theta}} \tag{23}$$

Since that term is constant with respect to $\boldsymbol{r'}$, the physical results should be the same. However, this is still a problem to be discussed because we expected that the charge should be $Q - q$ in order to have the total charge outside of the sphere (inclusive) be Q. Instead of $Q - q$ we find the image charge to be $2Q$.

This completes the calculation of the potential everywhere. Now we can start solving what we are asked to solve for:

(a) We can get this solution without using our result above, and then check it by using the potential we calculated. $U(\infty)$ is the potential energy when the point charge is infinitely far from the spherical charge. At that point, the two charges don't influence each other, so Q is uniformly distributed on the conductor. The total potential energy is equal to the work it takes to assemble that charge distribution. Then we go to $r = 0$. The charges are again uniformly distributed, so the total potential energy will be $U(\infty)$ plus the potential energy of the interaction. Griffiths [**?**] gives the formula for the total potential energy of a configuration:

$$U(r) - U(\infty) = \frac{1}{2}Q\frac{Q}{4\pi\epsilon_0 a}2 = \frac{Q^2}{4\pi\epsilon_0 a} \tag{24}$$

Now we can calculate the same thing using our potential from above. To get $U(\infty)$, look at the case $r > a$. The potential at r'=r is obtained by removing from the potential the term generated by the point charge at r and plugging in $r' = r$, $\theta = 0$.

$$V(r' = a, r = \infty) = \frac{Q}{4\pi\epsilon_0 a}$$

$$V(r' = r = \infty) = 0$$

$$U(\infty) = \frac{Q^2/2}{4\pi\epsilon_0 a}$$

To get $U(0)$, look at the case $r < a$.

$$V(r' = a, r = 0) = \frac{2Q}{4\pi\epsilon_0 a}$$

$$V(r' = r = 0) = \frac{Q}{4\pi\epsilon_0 a}$$

$$U(0) = \frac{1}{2}[\frac{2Q^2}{4\pi\epsilon_0 a} + \frac{Q^2}{4\pi\epsilon_0 a}] = \frac{3Q^2/2}{4\pi\epsilon_0 a}$$

$$U(0) - U(\infty) = \frac{Q^2}{4\pi\epsilon_0 a} \tag{25}$$

Notice that the two solutions match. However, there are two puzzles: firstly, when we move the point charge from infinity to the origin, the charges on the sphere will rearrange; doesn't that require more work, and hence increase the potential energy difference? Second, we'll see later that the potential energy difference will diverge at r=a. This seems to indicate that we can never move the charge from infinity to the origin, and hence $U(0) - U(\infty)$ doesn't make sense at all. To study this, let's study the potential energy difference for r close to a. For $\epsilon > 0$:

- $r = a(1+\epsilon)$:

$$V(r' = a) = \frac{Q(2-\epsilon)}{4\pi\epsilon_0 a}$$
$$V(r' = r) = -\frac{Q}{4\pi\epsilon_0 2a\epsilon}$$

- $r = a(1+\epsilon)$:

$$V(r' = a) = \frac{2Q}{4\pi\epsilon_0 a}$$
$$V(r' = r) = -\frac{Q}{4\pi\epsilon_0 2a\epsilon}$$

Now we note the following. For $r < a$:

$$U(r) - U(\infty) = [U(r) - U(a(1-\epsilon))] + [U(a(1-\epsilon)) - U(a(1+\epsilon)] + [U(a(1+\epsilon)) - U(\infty)]$$

The middle term will be zero because the divergences are equal, and thus we can have the potential energy difference between points inside and outside the sphere be well-defined.

(b) We found the potential at the sphere and at the point charge for r close to a, now we just have to get the potential energy from those. For $\epsilon > 0$:

$$U(a(1+\epsilon)) - U(\infty) = U(a(1-\epsilon)) - U(\infty) = -\frac{Q^2}{4\pi\epsilon_0 4a\epsilon} \tag{26}$$

I cannot make a sketch here, but the point is the graph goes to $-\infty$ from both sides at $r = a$, it becomes constant at $r = 0$, and it is positive and decreasing as $r \to \infty$.

(c) From (3),

$$V(r' = r > a) = \frac{Q(1 + a/r)}{4\pi\epsilon_0 r} - \frac{Qa}{4\pi\epsilon_0 (r^2 - a^2)} \tag{27}$$

$$V(r' = r >> a) \approx \frac{Q(1 + a/r)}{4\pi\epsilon_0 r} - \frac{Qa}{4\pi\epsilon_0 r^2(1 - \frac{a^2}{r^2}} \approx \frac{Q(1 + a/r)}{4\pi\epsilon_0 r} - \frac{Qa(1 + \frac{a^2}{r^2})}{4\pi\epsilon_0 r^2}$$

$$= \frac{Q}{4\pi\epsilon_0 r}(1 - \frac{a^3}{r^3}) \tag{28}$$

$$U(r) = \frac{1}{2}\frac{Q^2}{4\pi\epsilon_0 r}(1 - \frac{a^3}{r^3}) + \frac{1}{2}\frac{Q(1 + a/r)}{4\pi\epsilon_0 a}Q$$

$$U(r) - U(\infty) = \frac{Q^2}{4\pi\epsilon_0 r} - \frac{Q^2}{8\pi\epsilon_0 r}\frac{a^3}{r^3} \tag{29}$$

(d) From (27),

$$U(r > a) - U(\infty) = \frac{1}{2}\frac{Q^2(1 + a/r)}{4\pi\epsilon_0 r} - \frac{1}{2}\frac{Q^2 a}{4\pi\epsilon_0(r^2 - a^2)} + \frac{Q^2(1 + a/r)}{2 * 4\pi\epsilon_0 a} - \frac{Q^2}{2 * 4\pi\epsilon_0 a}$$

$$= \frac{Q^2}{4\pi\epsilon_0 r} + \frac{Q^2 a}{2 * 4\pi\epsilon_0 r^2} - \frac{Q^2 a}{2 * 4\pi\epsilon_0(r^2 - a^2)} \tag{30}$$

Similarly, from (23),

$$V(r = r' < a) = \frac{2Q}{4\pi\epsilon_0 a} - \frac{Qa}{4\pi\epsilon_0(a^2 - r^2)}$$

$$U(r < a) - U(\infty) = \frac{2Q^2}{2 * 4\pi\epsilon_0 a} - \frac{Q^2 a}{2 * 4\pi\epsilon_0(a^2 - r^2)} + \frac{2Q^2}{2 * 4\pi\epsilon_0 a} - \frac{Q^2}{2 * 4\pi\epsilon_0 a}$$

$$= \frac{3Q^2}{2 * 4\pi\epsilon_0 a} - \frac{Q^2 a}{2 * 4\pi\epsilon_0(a^2 - r^2)} \tag{31}$$