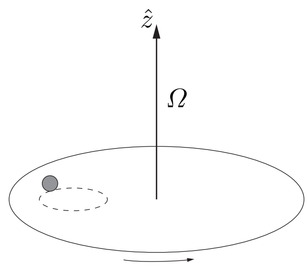
J07M.1 - Ball on a Turntable

Problem



A spherically symmetric ball of mass m, moment of inertia I about any axis through its center, and radius a, rolls without slipping and without dissipation on a horizontal turntable (so frictional forces act on the ball at its point of contact with the turntable, but do no work). The turntable is rotating about the vertical z-axis at constant angular velocity $\Omega\hat{\boldsymbol{z}}$.

Use a laboratory frame coordinate system $(\boldsymbol{r}, z), \boldsymbol{r} = (x, y)$ so the center of the ball is in the plane $z = 0$, and the axis of the turntable is $\boldsymbol{r} = 0$. Distinguish the (vector) angular velocity $\boldsymbol{\omega}_{rot}$ of the rotation of the ball about its center, and the angular velocity $\omega_{cm}\hat{\boldsymbol{z}}$ of the orbit about its center, where ω_{cm} is scalar.

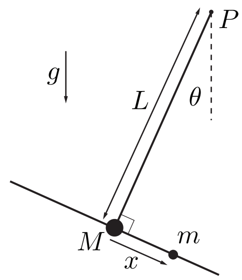
a) Find a solution of the equations of motion where the center of the ball is stationary at a point $\boldsymbol{r}_0 \neq 0$ in the laboratory frame. What is $\boldsymbol{\omega}_{rot}(t)$ for this solution?

b) Find the general solution for the orbit $\boldsymbol{r}(t)$ of the center of the ball, when the initial center-of-mass position and velocity are $\boldsymbol{r}_0$ and $\boldsymbol{v}_0$. Show that, in the laboratory frame, the ball rolls along a circular path (in general centered at a point $\boldsymbol{r} \neq 0$), with an angular velocity ω_{cm} that is independent of the initial conditions.

c) If the ball is solid with uniform density, what is the relation of ω_{cm} to Ω?

J07M.2 - Inverted T Pendulum

Problem



Two massless rods are rigidly joined at right angles in a rigid inverted "T" shape, with a point mass M attached to the junction. A second point mass m is free to slide frictionlessly along the first rod that forms the base of the inverted "T", as shown. (Its motion is not impeded by the mass M.)

This whole arrangement is suspended in Earth's gravity from a frictionless pivot P attached to the end of the second rod (see figure), where the distance from the pivot to the junction of the rods is L. Let θ be the angle of the second rod relative to the vertical, and x be the displacement of the mobile mass m relative to the fixed mass M, as shown.

a) Initially, the system is at rest with $\theta = 0$ and $x = 0$. Analyze the linearized equations of motion about this stationary state for small values of θ and x, to find the eigenmodes. For each eigenmode, determine whether it is stable or unstable.

b) At time $t = 0$ an instantaneous small horizontal impulse J is applied to the mass m, so that immediately afterwards ($t = 0^+$) it has finite velocity along the horizontal rod ($\dot{x} > 0$), but $\dot{\theta} = 0$. Solve the linearized equations of motion for the subsequent motion $x(t)$ and $\theta(t)$.

J07M.3 - Bubble in an Incompressible Fluid

Problem

An explosion at time $t = 0$ in an ideal (zero viscosity) incompressible fluid produces a perfectly spherically symmetric expanding bubble of vacuum with radius $R(t)$ (neglect the effect of any gas or vapor inside the bubble). The bubble expands to maximum radius R_{max} and then collapses. The pressure in the fluid far from the bubble is P_∞, and the mass density of the fluid is ρ. Neglect any effects of surface tension or gravity; assume the bubble remains spherically symmetric at all times, and that the velocity field in the fluid is purely radial.

a) Obtain an expression for the velocity field inside the fluid, and hence get an expression for the total energy (kinetic + potential) of the fluid in terms of R and dR/dt.

b) Obtain an equation of motion for the bubble's radius $R(t)$ of the form

$$\frac{dR}{dt} = f(R)$$

What is the function $f(R)$?

c) How long does it take for the bubble to collapse after it reaches its maximum radius? Your answer can contain a finite dimensionless integral whose value you have not obtained.

d) What is the asymptotic behavior of $R(t)$ in the final moments of the bubble's collapse when $R \ll R_{max}$? (Do **not** consider the possibility of "cold fusion"!)

J07E.1 - Point Charge and Conducting Sphere

Problem

A point charge Q is located at a distance r away from the center of a thin spherical *conducting* shell of radius a, which has a net charge also equal to Q. Let $U(r)$ be the total electrostatic potential energy of this system.

a) What is $U(0) - U(\infty)$?

b) Determine the leading behavior of $U(r) - U(\infty)$ as $r \to a$, and make a qualitatively correct sketch showing its important features over the whole range $0 \leq r < \infty$.

c) As $r \to \infty, U(r) - U(\infty) \to Q^2/4\pi\epsilon_0 r$. Obtain the leading correction to this behavior for large r.

d) If you have not already done so, give the explicit function $U(r) - U(\infty)$ for all r.

J07E.2 - Rotating Shell of Charge

Problem

A hollow spherical shell centered at the origin has radius a and a total electric charge $Q > 0$ unifromly distributed over its surface. The shell is slowly spun up to an angular velocity $\omega = \omega_0 \hat{z}$ (where $\omega_0 > 0$) over a period of time $\tau \gg a/c$, where c is the speed of light, so radiation effects can be ignored.

a) To linear order in $d\omega/dt$, find expressions for the electromagnetic fields $\vec{E}(\vec{r})$ and $\vec{B}(\vec{r})$ throughout space, as functions of ω and $d\omega/dt$. Make a qualitatively correct sketch showing the pattern of electric field lines in the plane $z = 0$. Indicate the direction of rotation of the charged shell on your plot.

b) After the angular velocity ω_0 is reached, hat is the total angular momentum $\vec{L}$ stored in the electromagnetic fields?

J07E.3 - Negative Dielectric Constant

Problem

Electromagnetic waves with (angular frequency ω can propagate without loss in a linear isotropic medium where the (frequency-dependent) dielectric constant $\epsilon(\omega)$ and relative magnetic permeability $\mu(\omega)$ are both real, and their product is positive. If ϵ and μ have no frequency dependence, electromagnetic stability requires $\epsilon > 0$ and $\mu > 0$, but if they are frequency-dependent, this condition becomes $(\omega\epsilon)' \equiv d(\omega\epsilon)/d\omega > 0$ and $(\omega\mu)' \equiv d(\omega\mu)/d\omega > 0$. This allows $\epsilon(\omega)$ and $\mu(\omega)$ to be negative at some frequencies, if they have strong frequency dependence.

While strong frequency dependence of ϵ is common, most materials have $\mu \simeq 1$ at optical frequencies. Recently, however, artificial structures where ϵ and μ are *both* negative in some frequency range have been created.

a) Solve the Maxwell equations for an electromagnetic wave in a unifrom medium that is isotropic and time-reversal invariant: $\epsilon(\omega)$ and $\mu(\omega)$ are real, even functions of ω in the frequency range of interest, and c is the speed of light in vacuum. For $\vec{X} = \vec{E}, \vec{D}, \vec{B}$ and $\vec{H}$, find the electromagnetic fields $\vec{X}(\vec{r}, t) = \vec{X}_0 \cos(\vec{k} \cdot \vec{r} - \omega t)$. Give the wavenumber $|\vec{k}| = k(\omega)$ as a function of (angular) frequency ω.

b) Obtain an expression for the ratio $(\omega/k)/v_g$, where $\omega/k(\omega)$ is the *phase velocity*, and $v_g(\omega)$ is the *group velocity*, the velocity at which energy propagates. Show that $v_g(\omega)$ is negative (and *antiparallel* to the phase velocity) if $\epsilon(\omega)$ and $\mu(\omega)$ are negative.

c) A narrow beam of light from a laser operating at frequency ω is incident on the flat surface of an isotropic medium with negative $\epsilon(\omega)$ and $\mu(\omega)$ (and hence negative $v_g(\omega)$) at an angle θ_i relative to the normal. The wavenumber of waves with frequency ω in the medium is $k(\omega)$. What is the angle θ_r made by the refracted beam in the medium? Make a diagram showing a possible path of the incident and refracted beams. Assume that the incoming beam is traveling in a vacuum. (You may find it helpful to consider the relation between the incident and refracted photon wavevectors $\vec{k}_i$ and $\vec{k}_r$.)

J07Q.1 - Excitation from a Delta Function Potential

Problem

Consider a non-relativistic mass m particle with coordinate x in one dimension that is subject to an attractive delta-function potential at $x = 0$, *i.e.*, a potential $V(x) = -V_0\delta(x/a)$, with $V_0 > 0$.

a) The ground state of the particle is a bound state. Find its wave function and binding energy.

b) The particle is now perturbed by a weak time dependent potential $V(x,t) = Fx\cos(\omega t)$. Find the transition rate from the bound state to the continuum. (It may help to confine the particle in a large box $|x| < L/2$ and then take the limit $L \to \infty$.)

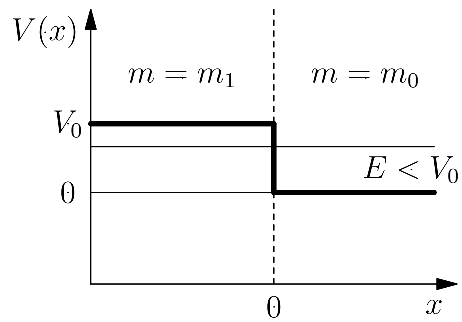
J07Q.2 - Effective Mass

Problem

An electron is moving in one dimension in a potential $V(x) = 0$ for $x > 0$ and $V(x) = V_0 > 0$ for $x < 0$. The region $x > 0$ is empty space, where the electron mass is the usual bare mass m_0, but in the region $x < 0$ it has a modified "effective mass" m_1. When the mass of a non-relativistic particele depends on its position, the Hamiltonian should be written in the operator-ordered form

$$H = \frac{1}{2}p(m(x))^{-1}p + V(x)$$

where $[x, p] = i\hbar$.



a) The standard continuity conditions (continuity of $\Psi(x)$ and $\Psi'(x) \equiv d\Psi(x)/dx$) only apply at $x = 0$ if $m_1 = m_0$. Derive the continuity conditions that apply at points where the mass is discontinuous.

b) The (unnormalized) wave function of an eigenstate of the Hamiltonian with an energy $E < V_0$ is given by $\Psi(x) = A \sin k(x - x_0)$ for $x > 0$. Find k, x_0 and $\Psi(x)$ for $x < 0$. Make a sketch of the function $\Psi(x)$, indicating its essential features.

J07Q.3 - Triangular Molecule

Problem

If a Na atom is represented by a singly occupied $3s$ orbital, the molecular orbitals of the triangular Na_3 *molecule* are given by the eigenstates $\psi_\nu(i) = \langle i|\psi_\nu\rangle$ of the one-electron Hamiltonian

$$h(\vec{t}) = \sum_{i=1}^{3} \epsilon_0 |i\rangle\langle i| - \sum_{i\neq j} t_{ij}|i\rangle\langle j|, \quad t_{ij} = t_{ji}, \quad \langle i|j\rangle = \delta_{ij}$$

with all t_{ij} real and positive, where $|i\rangle, i = 1,2,3$ is the $3s$ orbital on atom i. $\vec{t} = (t_{12}, t_{23}, t_{31})$ is a coordinate in a three-dimensional parameter space characterizing the molecular shape. For general values of $\vec{t}, h_{ij} = \langle i|h(\vec{t})|j\rangle$ is a generic 3×3 matrix, but when two or more of the t_{ij} are equal, symmetries of the eigenvectors make it easy to diagonalize (it is also easily diagonalized when any t_{ij} vanish).

a) FInd the three molecular orbitals $\psi_\nu(i)$ when $\vec{t} = (t, t, \lambda t)$. Classify them by symmetry, and sketch the energy levels $\epsilon_\nu(\lambda)$ as a function of $\lambda > 0$. How are they occupied in the Na_3 molecule ground state? What is special when $\lambda = 1$ (when all Na atoms are equivalent)?

The Born-Oppenheimer (BO) approximation determines molecular shape by minimizing the electronic energy as a function of fixed nuclear coordinates of the atoms. Your results in a) should imply that the BO energy has (at least) *three* distinct minima. Label them A,B and C. Tunneling of the molecular shape degrees of freedom between these minima restores the symmetry of the molecule, in which all three Na atoms are equivalent.

b) Model the molecular orbitals of A, B and C by those of $(t,0,0), (0,t,0)$ and $(0,0,t)$. The $\psi_\nu(i)$ can always be chosne real; if they change adiabatically as the shape evolves along the periodic path $A \to B \to C \to A, \psi_\nu(i) \to \eta_\nu\psi_\nu(i), \eta_\nu = \pm 1$. Determine η_ν for each orbital. (You may assume that the path from A to B passes through the configuration $(t/2, t/2, 0)$, and use this to fix the sign of $\psi_\nu(i)$ at B relative to that at A, *etc.*; only the *sign* ($\pm$ or 0) of $\psi_\nu(i)$ is important in this calculation.)

(Your results should imply that the BO ground state $|\psi_0\rangle$ of the Na_3 molecule evolves to $-|\psi_0\rangle$ along this path: this calculation was historically the first time that a "Berry Phase" was encountered.)

J07T.1 - Dilute Ionized Hydrogen

Problem

Consider an extremely dilute gas of partially ionized atomic hydrogen, such as occurred in the early universe. The binding energy of an electron and proton in the atomic ground state is $\epsilon \simeq 13.6$ eV. Let this dilute plasma be neutral, with equal numbers of electrons and protons. Assume it is at equilibrium at a temperature T such that $\epsilon/(k_B T) = 100$.

a) The fraction of the atoms ionized is 1/2, so the densities of atoms, free electrons, and free protons are all equal. What is this density? Give it as a formula in terms of T, ϵ, and any fundamental constants or particle masses, and then, for $\epsilon/k_B T = 100$, give a numerical density, in units of m$-$3, correct to the nearest order of magnitude Make and justify any appropriate approximations that will simplify your calculation.

b) At this density, estimate how much you need to lower the temperature to reduce the fraction of ionized atoms to 1/10, so only 10 % of the electrons are free, while the remainder are bound in atoms. Again, make and justify any reasonable approximations

Useful constants: $\hbar c = 2000$ eV-Å $= 200$ eV-nm; $m_e c^2 \simeq 500,000$ eV.

J07T.2 - Diatomic Gas in an Electric Field (M06T.2)

Problem

Consider a dilute gas of diatomic molecules with number density n. Each molecule has a constant electric dipole moment μ. The temperature T is high enough so all degrees of freedom may be treated classically, and the correlations between molecules may be neglected.

a) Calculate, to leading non-vanishing order in the density n, the electric polarization density $\vec{P}$ in an arbitrary external electric field $\vec{E}$.

b) What is the dielectric constant ϵ of this gas?

J07T.3 - Spin Waves

Problem

Consider spin waves in an isotropic ferromagnetically ordered crystal. These are waves in which the spins on each atom oscillate in space and time. Just as with sound waves, the spin waves can be quantized and they can store internal energy in a crystal lattice. However, these waves have a different relation between frequency and wavenumber than do sound waves. In particular, at low wavenumber.

$$\omega(k) = Ak^2$$

where A is a constant. Consider a crystal containing N spins in thermal equilibrium at temperature T.

a) What is the average energy in a spin wave mode of frequency ω? (Neglect the zero-point energy of the mode).

b) At low temperatures, the heat capacity of the spin wave system in the crystal is proportional to T^α. What is the numerical value of α?

c) If the material is a metal, do the spin waves give the dominant contribution to the heat capacity in the low-temperatue limit? What if the material is an insulator? Explain both of your answers.