

# 1 Thermodynamics

## 1.1 Problem 1

In this problem we have to be very careful with the assumptions we make. First of all, we note that the energy levels of a harmonic oscillator are:

$$\epsilon_n = \left( n + \frac{1}{2} \right) \hbar\omega$$

where  $n = n_x + n_y + n_z$ . However, for our purposes the ground state energy will be an annoyance, and as explained in Appendix A of Daniel V. Schroeder's "Thermal Physics", it doesn't really matter whether we include it or not (unless we get into very technical quantum details at the Planck length and stuff like that. We henceforth declare:

$$\epsilon_n = n\hbar\omega$$

(a) We have to get the number of ways to sum three numbers and get a certain result. Fix  $n$ , then you can choose  $n_x$  to be whatever you want between 0 and  $n$ . That, in turn, determines  $n_y$  to be between 0 and  $n - n_x$ . After that  $n_z$  is fixed at the value  $n - n_x - n_y$ . Thus, the total number of states with sum  $n$  is:

$$\begin{aligned} D(n) &= \sum_{n_x=0}^n \sum_{n_y=0}^{n-n_x} 1 = \sum_{n_x=0}^n (n - n_x + 1) = (n+1)(n+1) - \frac{n(n+1)}{2} \\ &= (n+1) \left( n+1 - \frac{n}{2} \right) = (n+1) \left( \frac{n}{2} + 1 \right) = \frac{n^2}{2} + \frac{3n}{2} + 1 \\ &\quad \left( \frac{n^2}{2} + \frac{3n}{2} + 1 \right) dn = D(\epsilon) d\epsilon \\ &\quad \left[ \frac{\epsilon^2}{2(\hbar\omega)^2} + \frac{3\epsilon}{2\hbar\omega} + 1 \right] \frac{1}{\hbar\omega} d\epsilon = D(\epsilon) d\epsilon \\ D(\epsilon) &= \begin{cases} \frac{\epsilon^2}{2(\hbar\omega)^3} + \frac{3\epsilon}{2(\hbar\omega)^2} + \frac{1}{\hbar\omega} & (exact) \\ \frac{\epsilon^2}{2(\hbar\omega)^3} & \epsilon \gg \hbar\omega \end{cases} \end{aligned} \quad (1)$$

(b) The no-assumptions way to determine the chemical potential is implicitly through the equation:

$$N = \sum_n D(n) \frac{1}{e^{(\epsilon_n - \mu)/kT} - 1}$$

ASSUMPTION:  $kT \ll \hbar\omega$  At low enough temperatures, basically all the particles will be in the ground state, so we can make the assumption:

$$N \approx D(0) \frac{1}{e^{(\epsilon_0 - \mu)/kT} - 1} = \frac{1}{e^{-\mu/kT} - 1}$$

Since  $N$  is large, the denominator must be small, the exponential must be close to 1, and therefore we can expand it to first order:

$$N \approx \frac{1}{1 - \mu/kT - 1} = \frac{kT}{-\mu}$$

$$\mu = -\frac{kT}{N} \quad (2)$$

The same answer is obtained if we keep the first excited term too.

(c) Let's get back to the equation for  $N$ :

$$N = \sum_n D(n) \frac{1}{e^{(\epsilon_n - \mu)/kT} - 1}$$

ASSUMPTION:  $kT \gg \hbar\omega$  This is told in the problem, and it allows us to change this sum into an integral:

$$N = \int_0^\infty D(\epsilon) \frac{1}{e^{(\epsilon - \mu)/kT} - 1} d\epsilon$$

$$= \int_0^\infty \left[ \frac{\epsilon^2}{2(\hbar\omega)^3} + \frac{3\epsilon}{2(\hbar\omega)^2} + \frac{1}{\hbar\omega} \right] \frac{1}{e^{(\epsilon - \mu)/kT} - 1} d\epsilon$$

Now, the number of particles in the ground state is:

$$N_0 = \frac{1}{e^{-\mu/kT} - 1}$$

ASSUMPTION:  $T$  is low enough that  $N_0$  is quite large Under this assumption we might guess that the exponential is very close to 1, and we get:

$$\mu \approx -kT/N_0$$

But since  $N_0$  is quite large, we can take a guess that  $\mu \approx 0$ , and plug that into the integral:

$$N \approx \int_0^\infty \left[ \frac{\epsilon^2}{2(\hbar\omega)^3} + \frac{3\epsilon}{2(\hbar\omega)^2} + \frac{1}{\hbar\omega} \right] \frac{1}{e^{\epsilon/kT} - 1} d\epsilon$$

Change variables to  $x = \epsilon/kT$ . Then,

$$N = \int_0^\infty \left[ \frac{x^2(kT)^2}{2(\hbar\omega)^3} + \frac{3xkT}{2(\hbar\omega)^2} + \frac{1}{\hbar\omega} \right] \frac{1}{e^x - 1} kT dx$$

We can rewrite it as:

$$N = \frac{(kT)^3}{2(\hbar\omega)^3} \int_0^\infty x^2 \frac{1}{e^x - 1} dx + \frac{3(kT)^2}{2(\hbar\omega)^2} \int_0^\infty \frac{x}{e^x - 1} dx + \frac{kT}{\hbar\omega} \int_0^\infty \frac{1}{e^x - 1} dx$$

Each of these integrals is just a constant, but the first one will dominate in the limit  $kT \gg \hbar\omega$ . Define:

$$I = \int_0^{\infty} \frac{x^2}{e^x - 1} dx$$

then,

$$N = \frac{(kT)^3}{2(\hbar\omega)^3} I$$

This is only valid at one temperature, for the left hand side of the equation does not contain any T-dependence. We call this temperature  $T_E$ :

$$T_E = \frac{(2N)^{1/3} \hbar\omega}{I^{1/3} k} \quad (3)$$

Now notice that while the integrand in terms of  $\epsilon$  was 0 at  $\epsilon = 0$ , we know from a previous computation (and we assumed it in this part of the problem), that  $N_0$  is very large. Thus, we are not including the lowest state occupation in our calculation. We therefore can say that the number of excited states is:

$$N_e \approx \frac{(kT)^3}{2(\hbar\omega)^3} I$$
$$N_e = N \left( \frac{T}{T_E} \right)^3$$

This gives us our final result:

$$\alpha = 3 \quad (4)$$