1 January 2006, Quantum Mechanics, Problem 2

1.1 (a)

Case $g = 0$:

There is no interaction, so the particles move under:

$$-\frac{\hbar^2}{2m}\Psi = E\Psi$$

The lowest energy eigenstate that satisfies periodic boundary conditions will be:

$$\Psi = \frac{1}{\sqrt{L}} \tag{1}$$

$$E = 0 \tag{2}$$

This state has no degeneracy.

Case $g \neq 0$:

The strategy is to separate the solutions into a part depending on $\eta = \frac{x_1+x_2}{2}$, and another part depending on $x = x_1 - x_2$. These variables have corresponding momenta p_η and p, and the hamiltonian is:

$$H = \frac{p_\eta^2}{4m} + \frac{p^2}{m} + g\delta(x)$$

The first part will be a constant because this will minimize the energy (it can be seen from separation of variables). The second part is more challenging. I looked at two types of solutions: exponentials and sinusoidals. The ultimate problem with exponentials was that they could not satisfy periodic boundary conditions. So after struggling with a linear combination of sines and cosines, I tried instead this other form:

$$X_l = Acos(Bx + C)$$

$$X_r = Dcos(Ex + F)$$

where "l" and "r" denote the solutions for $x \leq 0$ and $x \geq 0$, respectively, $x = x_1 - x_2$ and $X(x)$ is the part of the wavefunction that depends only on this variable. The whole solution is written down for the interval $-L/2 \leq x \leq L/2$. We have to impose four conditions:

(i) Continuity of the function and discontinuity of its derivative at $x = 0$.

(ii) Normalization.

(iii) Periodicity.

(iv) Symmetry.

The last is due to the bosonic nature of the particles. The solution turns out to be:

$$\Psi(x) = Acos(\frac{2C|x|}{L} - C) \tag{3}$$

We have to determine C and A through the aforementioned conditions:

(i) The continuity of the function at $x = 0$ is automatically satisfied. Its derivatives are:

$$\Psi'_r(0) = \frac{2CA}{L} sinC$$

$$\Psi'_l(0) = -\frac{2CA}{L} sinC$$

$$-\frac{\hbar^2}{m}\left(\frac{4CA}{L} sinC\right) + gAcosC = 0$$

$$\frac{4C\hbar^2}{mLg} tanC = 1 \tag{4}$$

This equation defines C.

(ii) Normalization gives an equation that serves to define A:

$$\frac{A^2 L}{2}\left(1 + \frac{sin(2C)}{2C}\right) = 1 \tag{5}$$

(iii) The continuity at the ends is satisfied automatically by the evenness of the function. As for the derivative, it will contain a $sin(C - C) = 0$, so it will be 0 at the ends and therefore continuous. The function is thus differentiable at the endpoints, so it satisfies periodic boundary conditions.

(iv) The function is even, therefore symmetric under exchange of the bosons.

The energy of this ground state is found by applying the Hamiltonian operator:

$$E = \frac{1}{m}\left(\frac{2C\hbar}{L}\right)^2 \tag{6}$$

The state has no degeneracy even though the equation for C has two solutions, simply because those two solutions give the same state (cos(q)=cos(-q)).

1.2 (b)

The bosonic states must be symmetric. The ground state will thus have a symmetric spin state (because the space ground state is symmetric and all spin states are degenerate). By the usual raising and lowering trick we can find all the states of definite total spin and z-component of total spin in terms of the states of definite z-component of each spin. We denote the z-component of each spin with "↑", "→" or "↓". The states are:

$$|22>= |\uparrow\uparrow>$$

$$|21>= \frac{|\rightarrow\uparrow> + |\uparrow\rightarrow>}{\sqrt{2}}$$

$$|20>= \frac{|\downarrow\uparrow> + 2|\rightarrow\rightarrow> + |\uparrow\downarrow>}{\sqrt{6}}$$

$$|2-1>=\frac{|\downarrow\rightarrow>+|\rightarrow\downarrow>}{\sqrt{2}}$$

$$|2-2>=|\downarrow\downarrow>$$

$$|11>=\frac{-|\rightarrow\uparrow>+|\uparrow\rightarrow>}{\sqrt{2}}$$

$$|10>=\frac{-|\downarrow\uparrow>+|\uparrow\downarrow>}{\sqrt{2}}$$

$$|1-1>=\frac{-|\downarrow\rightarrow>+|\rightarrow\downarrow>}{\sqrt{2}}$$

$$|00>=\frac{|\downarrow\uparrow>-|\rightarrow\rightarrow>+|\uparrow\downarrow>}{\sqrt{3}}$$

Notice that all the states of total spin 0 or 2 are symmetric, while all the states of total spin 1 are antisymmetric. So the ground state wavefunction is one of the following:

$$\Psi(x)=Acos(\frac{2C|x|}{L}-C)|22> \tag{7}$$

$$\Psi(x)=Acos(\frac{2C|x|}{L}-C)|21> \tag{8}$$

$$\Psi(x)=Acos(\frac{2C|x|}{L}-C)|20> \tag{9}$$

$$\Psi(x)=Acos(\frac{2C|x|}{L}-C)|2-1> \tag{10}$$

$$\Psi(x)=Acos(\frac{2C|x|}{L}-C)|2-2> \tag{11}$$

$$\Psi(x)=Acos(\frac{2C|x|}{L}-C)|00> \tag{12}$$

$$E=\frac{1}{m}\left(\frac{2C\hbar}{L}\right)^2 \tag{13}$$

The degeneracy is 6, and the total spin is 2 for the first 5 functions and 0 for the last one. The first excited space state will be antisymmetric, so the spin part will have to be antisymmetric as well. Thus we will have to choose one of the other spin functions. The total spin will be 1 and the degeneracy will be 3.