J06Q.2

Solution to J06Q.2 — Two Indistinguishable Bosons

Part a

First change variables to the center of mass frame

$$\bar{x} = \frac{x_1 + x_2}{2} \qquad (1)$$
$$x = x_1 - x_2$$

Making the Hamiltonian

$$H = \frac{p^2 + 4\bar{p}^2}{m} + g\delta(x) \qquad (2)$$

In the center of mass frame $\bar{p} = 0$ so we consider just the Hamiltonian

$$H = \frac{p^2}{m} + g\delta(x) \qquad (3)$$

This is analogous to a 1-D delta potential problem, but since we are dealing with a ring we impose periodic boundary at $x = \pm\frac{L}{2}$.

We consider solutions on each side of the delta function.

$$\psi_L = A_L \cos(\omega_L x + \phi_L) \qquad (4)$$
$$\psi_R = A_R \cos(\omega_R x + \phi_R)$$

Since we are dealing with bosons, we know that the overall wavefunction must be symmetric:

$\psi_L(-x) = \psi_R(x)$. This imposes constraints on our wavefunctions, leaving us with

$$\begin{aligned} \psi_L &= A\cos(-\omega x + \phi) \\ \psi_R &= A\cos(\omega x + \phi) \end{aligned} \tag{5}$$

Continuity of the first derivative at $x = \pm\frac{L}{2}$ gives

$$\begin{aligned} -\omega A\sin(\frac{\omega L}{2} + \phi) &= \omega A\sin(\frac{\omega L}{2} + \phi) \\ \frac{\omega L}{2} + \phi &= 0 \\ \omega &= -\frac{2\phi}{L} \end{aligned} \tag{6}$$

Discontinuity of the derivative over the delta function provides an additional constraint

$$\begin{aligned} \frac{\hbar^2}{m}(A\omega\sin\phi + A\omega\sin\phi) &= gA\cos\phi \\ \tan\phi = \frac{mg}{2\omega\hbar^2} &= \frac{-mgL}{4\phi\hbar^2} \end{aligned} \tag{7}$$

We can get $A = \sqrt{\frac{2}{L}}$ by normalizing, leaving us with our full solution of the ground state wave function:

$$\begin{aligned} &\boxed{\psi = \sqrt{\tfrac{2}{L}}\cos(|\omega|x + \phi)} \\ &\boxed{\tan\phi = \tfrac{-mgL}{4\phi\hbar^2}} \end{aligned} \tag{8}$$