Let's change notation so that the problem reduces to the following. Suppose we have a harmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \tag{1}$$

Hamiltonia

and eigenfunctions $\phi_n(x)$. If at $t = 0$ the wavefunction is

$$\psi(x, 0) = e^{-ipx_0}\phi_0(x) = \phi_0(x - x_0)\,, \tag{2}$$

PsiZero

what is $\psi(x, t)$? (I assumed $\hbar = 1$.)

Since the time evolution operator is e^{iHt}, we need to compute $\psi(x, t) = e^{iHt}e^{-ipx_0}\phi_0(x)$. This quantity can be easily computed using the Heisenberg picture as an intermediate step. In the Heisenberg picture, the equations of motion for $x(t)$ and $p(t)$ are

$$p(t) = m\dot{x}(t) \qquad \dot{p}(t) = -m\omega^2 x(t)\,. \tag{3}$$

xpeoms

The solution of (3) that satisfies $p(0) = p$ and $x(0) = x$ is

$$x(t) = x\cos\omega t + \frac{p}{m\omega}\sin\omega t \qquad p(t) = -m\omega x\sin\omega t + p\cos\omega t\,. \tag{4}$$

xpSoln

Here, $x(t)$ and $p(t)$ are Heisenberg operators, while x and p are Schrödinger operators. It follows that

$$\begin{aligned} e^{iHt}e^{-ipx_0}e^{-iHt} &= e^{-ip(t)x_0} = e^{-ix_0(p\cos\omega t - xm\omega\sin\omega t)} \\ &= e^{ixx_0 m\omega\sin\omega t}e^{-ipx_0\cos\omega t}e^{-\frac{i}{4}x_0^2 m\omega\sin(2\omega t)}\,. \end{aligned} \tag{5}$$

TransEOM

Hence

$$\begin{aligned} \psi(x, t) &= e^{iHt}e^{-ipx_0}e^{-iHt}e^{\frac{i}{2}\omega t}\phi_0(x) \\ &= e^{ixx_0 m\omega\sin\omega t}e^{-ipx_0\cos\omega t}e^{-\frac{i}{4}x_0^2 m\omega\sin(2\omega t)}e^{\frac{i}{2}\omega t}\phi_0(x) \\ &= e^{\frac{i}{2}\omega t}e^{-\frac{i}{4}x_0^2 m\omega\sin(2\omega t)}e^{ixx_0 m\omega\sin\omega t}\phi_0(x - x_0\cos\omega t)\,. \end{aligned} \tag{6}$$

MoreTimeDe

It is clear that one can replace $\phi_0(x)$ by $\phi_n(x)$ throughout and everything still holds with the only difference that $e^{\frac{i}{2}\omega t}$ in (6) is replaced by $e^{i\omega t(n+1/2)}$.