Let’s change notation so that the problem reduces to the following. Suppose we have a harmonic oscillator with Hamiltonian

\[ H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 \]  

(1)  

and eigenfunctions \( \phi_n(x) \). If at \( t = 0 \) the wavefunction is \( \psi(x, 0) = e^{-ipx_0}\phi_0(x) = \phi_0(x - x_0) \), what is \( \psi(x, t) \)? (I assumed \( \hbar = 1 \).)

Since the time evolution operator is \( e^{iHt} \), we need to compute \( \psi(x, t) = e^{iHt}e^{-ipx_0}\phi_0(x) \). This quantity can be easily computed using the Heisenberg picture as an intermediate step. In the Heisenberg picture, the equations of motion for \( x(t) \) and \( p(t) \) are

\[ p(t) = m\dot{x}(t) \quad \dot{p}(t) = -m\omega^2x(t). \]  

(3)  

The solution of (3) that satisfies \( p(0) = p \) and \( x(0) = x \) is

\[ x(t) = x \cos \omega t + \frac{p}{m\omega} \sin \omega t \quad p(t) = -m\omega x \sin \omega t + p \cos \omega t. \]  

(4)  

Here, \( x(t) \) and \( p(t) \) are Heisenberg operators, while \( x \) and \( p \) are Schrödinger operators. It follows that

\[ e^{iHt}e^{-ipx_0}e^{-iHt} = e^{-ip(t)x_0} = e^{-ix_0(p \cos \omega t - x \sin \omega t)} \]

\[ = e^{ixx_0m \sin \omega t}e^{-ipx_0 \cos \omega t}e^{-\frac{1}{2}x_0^2m \sin(2\omega t)} \]  

(5)  

Hence

\[ \psi(x, t) = e^{iHt}e^{-ipx_0}e^{-iHt}e^{i\omega t}\phi_0(x) \]

\[ = e^{ixx_0m \sin \omega t}e^{-ipx_0 \cos \omega t}e^{-\frac{1}{2}x_0^2m \sin(2\omega t)}e^{i\omega t}\phi_0(x) \]

\[ = e^{i\omega t}e^{-\frac{1}{2}x_0^2m \sin(2\omega t)}e^{ixx_0m \sin \omega t}e^{i\omega t}\phi_0(x - x_0 \cos \omega t). \]  

(6)  

It is clear that one can replace \( \phi_0(x) \) by \( \phi_n(x) \) throughout and everything still holds with the only difference that \( e^{i\omega t} \) in (6) is replaced by \( e^{i\omega t(n+1/2)} \).