1 January 2006, Quantum Mechanics, Problem 1

1.1 (a)

The first part was done by everyone, if you need to see a proof ask me:

$$\frac{d<\boldsymbol{r}>}{dt} = \frac{<\boldsymbol{p}>}{m} \tag{1}$$

$$\frac{d<\boldsymbol{p}>}{dt} = -<\nabla V> \tag{2}$$

1.2 (b)

Here there were a few solutions that I've seen. Hans suggested assuming that the field is small and solving to second order. I didn't think that was reasonable because they always say it in the statement when the field is supposed to be small. Then I heard some other solution, in which something was left as an infinite sum, or something like that. I didn't like that either. Finally, I came across a third solution, which is no more satisfactory in the sense that it requires some non-trivial assumption, but the final answer is very satisfying (to me). By steps:

Assumption 1. *The final state is a coherent state, and hence its wave function is:*

$$\Psi = Be^{iP_0X}e^{-(X-X_0)^2}$$

where we are working with dimensionless units $X = \sqrt{\frac{m\omega}{\hbar}}x$ and $P = \frac{1}{\sqrt{m\hbar\omega}}p$.

1) Prove $<X>=X_0$. This is done by writing down the integral and noting that it is of the form $\int XGaussian(X-X_0)$, and thus is X_0 by parity arguments.

2) Prove $<P>=P_0$. This is done by writing down $<\Psi|p|\Psi$ and using the explicit form of the operator $\hat{P}$ and the previous part.

3) Compute $<X>$ and $<P>$ by using part (a) of the problem.

4) Plug these values into X_0 and P_0 above, then check that it satisfies the initial condition that at time $t=0$ the state should be the ground state of the harmonic oscillator with no electric field.

The final result is:

$$\Psi(x,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{iqEsin(\omega t)x/m\omega\hbar}e^{-\{x-\frac{qE}{m\omega^2}[1-cos(\omega t)]\}^2 m\omega/2\hbar} \tag{3}$$

As you may have guessed, the part I don't like is the assumption that the final state is a coherent state, because then you have to know this beforehand, and it is not trivial that one will.

The alternative way to solve this problem is like this:

$$|\Psi(x,t)>=\sum_n \alpha_{n^+}e^{-iE_{n^+}t/\hbar}|n^+>$$

where $^+$ denotes "after turning on the electric field", not to be confused with the dagger that goes on the creation operator.

$$|\Psi(x,0)>=\sum_n \alpha_{n^+}|n^+>=|0>$$

$$\hat{T}(x_0)\sum_n \alpha_{n^+}|n^+>=\hat{T}(x_0)|0>=|0^+>$$

where $x_0=\frac{-qE}{m\omega^2}$, the amount by which the origin is shifted, and $\hat{T}$ is the translation operator in position space.

$$\sum_n \alpha_{n^+}|n^+>=\hat{T^{-1}}(x_0)|0^+>=\hat{T}(-x_0)|0^+>$$

$$<m^+|\sum_n \alpha_{n^+}|n^+>=<m^+|\alpha_{m^+}|m^+>=\alpha_{m^+}=<m^+|\hat{T}(-x_0)|0^+>$$

$$\alpha_{m^+}=<m^+|e^{ipx_0/\hbar}|0^+>=<m^+|e^{(-a^++a)x_0\sqrt{m\omega/2\hbar}}|0^+>$$

$$=<m^+|e^{-x_0\sqrt{m\omega/2\hbar}a^+}e^{x_0\sqrt{m\omega/2\hbar}a}e^{-m\omega x_0^2/4\hbar}|0^+>$$

$$=<m^+|e^{-x_0\sqrt{m\omega/2\hbar}a^+}|0^+>e^{-m\omega x_0^2/4\hbar}=<m^+|\sum_k\left(-x_0\sqrt{\frac{m\omega}{2\hbar}}\right)^k\frac{(a^+)^k}{k!}|0^+>$$

$$=<m^+|\left(-x_0\sqrt{\frac{m\omega}{2\hbar}}\right)^m\frac{(a^+)^m}{m!}|0^+>=\left(-x_0\sqrt{\frac{m\omega}{2\hbar}}\right)^m\frac{1}{\sqrt{m!}}$$

$$|\Psi(x,t)>=\sum_n\left(-x_0\sqrt{\frac{m\omega}{2\hbar}}\right)^n\frac{1}{\sqrt{n!}}e^{-i\left(\left(n+\frac{1}{2}\right)\hbar\omega-\frac{q^2E^2}{2m\omega^2}\right)t/\hbar}|n^+>$$

$$|\Psi(x,t)>=\sum_n\left(-x_0\sqrt{\frac{m\omega}{2\hbar}}\right)^n\frac{1}{\sqrt{n!}}e^{-i\left(\left(n+\frac{1}{2}\right)\hbar\omega-\frac{q^2E^2}{2m\omega^2}\right)t/\hbar}\frac{(a^+)^n}{\sqrt{n!}}|0^+>$$

$$\Psi(x,t)=e^{\left(-x_0\sqrt{\frac{m\omega}{2\hbar}}\right)e^{-i\omega t}a^+}e^{-i\left(\frac{\hbar\omega}{2}-\frac{q^2E^2}{2m\omega^2}\right)t/\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-m\omega\left(x+\frac{qE}{m\omega^2}\right)^2/2\hbar}$$

$$e^{ka^+}=e^{k(a^++a-a)}=e^{k(a^++a)}e^{-ka}e^{-\frac{k^2}{2}}$$

$$e^{ka^+}=e^{k(a^++a-a)}=e^{k\sqrt{\frac{2m\omega}{\hbar}}(x+\frac{qE}{m\omega^2})}e^{-ka}e^{-\frac{k^2}{2}}$$

$$\Psi(x,t)=e^{-x_0\frac{m\omega}{\hbar}e^{-i\omega t}(x+\frac{qE}{m\omega^2})}e^{-x_0^2\frac{m\omega}{4\hbar}e^{-2i\omega t}}e^{-i\left(\frac{\hbar\omega}{2}-\frac{q^2E^2}{2m\omega^2}\right)t/\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-m\omega\left(x+\frac{qE}{m\omega^2}\right)^2/2\hbar} \quad (4)$$

This is a perfectly valid final answer, since it is in terms of x and t and known constants and functions. It would be nice, however, to check that it is the same as the previous answer. Also, the initial condition:

$$\Psi(x,0)=e^{\frac{qE}{\hbar\omega}(x+\frac{qE}{m\omega^2})}e^{-\left(\frac{qE}{m\omega^2}\right)^2\frac{m\omega}{4\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-m\omega x^2/2\hbar}e^{-q^2E^2/2\hbar m\omega^3}e^{-xqE/\hbar\omega}$$

$$\Psi(x,0)=e^{\frac{q^2E^2}{4m\hbar\omega^3}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-m\omega x^2/2\hbar}$$

This just tells us that we made some mistake with a factor of 2, but we'll ignore that because the spirit of the solution is overall fine. If you actually check the two solutions against each other, they are not equal. I don't see the reason for this, so I was hoping to be enlightened by one of you.

Silviu provided yet another answer relying on Heisenberg picture operators. This solution gave a new answer, different from the other two. Clearly this problem leaves lots of things open to interpretation...