J06Q.3 - Magnetic Resonance (solution by Jim Wu)
A particle of spin $\frac{1}{2}$ and magnetic moment μ is at rest in the time-dependent magnetic field

$$\vec{B} = B_0\hat{z} + B_1\hat{x}\cos\omega t - B_1\hat{y}\sin\omega t$$

which is often employed in magnetic resonance experiments. If the particle has the z component of its spin up (pointing along the positive z direction) at time $t = 0$, what is the probability that a measurement will find the z component of its spin down at time $t > 0$.

Solution:

The Hamiltonian of the system is the interaction between the magnetic moment and the external magnetic field:

$$H = -\vec{\mu}\cdot\vec{B} = -\frac{gq}{2m}\vec{S}\cdot\vec{B} = -\frac{gq}{2m}\frac{\hbar}{2}\vec{\sigma}\cdot\vec{B} = \frac{\hbar}{2}\left(-\omega_0\hbar\sigma_z + \omega_1\sigma_x\cos\omega t - \omega_1\sigma_y\sin\omega t\right)$$

Written in the S_z basis,

$$H \to \frac{\hbar}{2}\begin{pmatrix} -\omega_0 & \omega_1(\cos\omega t + i\sin\omega t) \\ \omega_1(\cos\omega t - i\sin\omega t) & \omega_0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} -\omega_0 & \omega_1 e^{i\omega t} \\ \omega_1 e^{-i\omega t} & \omega_0 \end{pmatrix}$$

Let's say that in the S_z basis, our wave function is

$$\psi \to \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$

Then from the time-dependent Schrödinger equation,

$$\frac{1}{2}\begin{pmatrix} -\omega_0 & \omega_1 e^{i\omega t} \\ \omega_1 e^{-i\omega t} & \omega_0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = i\begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix}$$

Notice that if $t = 0$, then we are left with uncoupled ODEs

$$\frac{1}{2}\begin{pmatrix} -\omega_0 & 0 \\ 0 & \omega_0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = i\begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix}$$

suggesting solutions of the form $a(t) = c(t)e^{i\omega_0 t/2}$ and $b(t) = d(t)e^{-i\omega_0 t/2}$. Substituting this back into the coupled ODE and computing the matrix multiplication yields

$$\begin{pmatrix} -\omega_0 c e^{i\omega_0 t/2} + \omega_1 e^{i\omega t} d e^{-i\omega_0 t/2} \\ \omega_1 e^{-i\omega t} c e^{i\omega_0 t/2} + \omega_0 d e^{-i\omega_0 t/2} \end{pmatrix} = 2i\begin{pmatrix} \frac{i\omega_0}{2} c e^{i\omega_0 t/2} + \dot{c} e^{i\omega_0 t/2} \\ -i\frac{\omega_0}{2} d e^{-i\omega_0 t/2} + \dot{d} e^{-i\omega_0 t/2} \end{pmatrix}$$
$$\begin{pmatrix} \omega_1 e^{i\omega t} d e^{-i\omega_0 t/2} \\ \omega_1 e^{-i\omega t} c e^{i\omega_0 t/2} \end{pmatrix} = 2i\begin{pmatrix} \dot{c} e^{i\omega_0 t/2} \\ \dot{d} e^{-i\omega_0 t/2} \end{pmatrix}$$
$$\begin{pmatrix} \omega_1 e^{i(\omega-\omega_0)t} d \\ \omega_1 e^{-i(\omega-\omega_0)t} c \end{pmatrix} = 2i\begin{pmatrix} \dot{c} \\ \dot{d} \end{pmatrix}$$

Now, taking a derivative of the final equation, we can finally decouple the differential equations

$$\begin{aligned}\begin{pmatrix}\ddot{c}\\ \ddot{d}\end{pmatrix} &= -\frac{i\omega_1}{2}\begin{pmatrix}\dot{d}e^{i(\omega-\omega_0)t} + i(\omega-\omega_0)de^{i(\omega-\omega_0)t}\\ \dot{c}e^{-i(\omega-\omega_0)t} - i(\omega-\omega_0)ce^{-i(\omega-\omega_0)t}\end{pmatrix}\\ &= -\frac{i\omega_1}{2}\begin{pmatrix}-\frac{i\omega_1}{2}c + i(\omega-\omega_0)(\frac{2i}{\omega_1}\dot{c})\\ -\frac{i\omega_1}{2}d - i(\omega-\omega_0)(\frac{2i}{\omega_1}\dot{d})\end{pmatrix}\\ &= \begin{pmatrix}-\frac{\omega_1^2}{4}c + i(\omega-\omega_0)\dot{c}\\ -\frac{\omega_1^2}{4}d - i(\omega-\omega_0)\dot{d}\end{pmatrix}\end{aligned}$$

These are exactly the same as the classical harmonic oscillator equations with damping. The solution to the two equations are

$$\begin{aligned}c(t) &= Ae^{\frac{i}{2}[(\omega-\omega_0)+\Omega]t} + Be^{\frac{i}{2}[(\omega-\omega_0)-\Omega]t}\\ d(t) &= Ce^{\frac{i}{2}[-(\omega-\omega_0)+\Omega]t} + De^{\frac{i}{2}[-(\omega-\omega_0)-\Omega]t}\end{aligned}$$

where $\Omega = \sqrt{\omega_1^2 + (\omega-\omega_0)^2}$. Therefore, the solution to the original problem is

$$\begin{aligned}a(t) &= c(t)e^{i\omega_0 t/2} = Ae^{\frac{i}{2}(\omega+\Omega)t} + Be^{\frac{i}{2}(\omega-\Omega)t}\\ b(t) &= d(t)e^{-i\omega_0 t/2} = Ce^{-\frac{i}{2}(\omega-\Omega)t} + De^{-\frac{i}{2}(\omega+\Omega)t}\end{aligned}$$

At $t = 0$, the particle was in the spin-up configuration and so $a(0) = 1$ and $b(0) = 0$. From Schrödinger's equation, this also suggests that $\dot{a}(0) = \frac{i\omega_0}{2}$.

For $a(t)$, we have

$$\begin{cases}A + B = 1\\ \frac{i}{2}(\omega+\Omega)A + \frac{i}{2}(\omega-\Omega)B = i\frac{\omega_0}{2}\end{cases} \quad\Rightarrow\quad \begin{cases}A + B = 1\\ \omega(A+B) + \Omega(A-B) = \omega_0\end{cases}$$

which implies that $A = \frac{1}{2}\left(1 + \frac{\omega_0-\omega}{\Omega}\right)$ and $B = \frac{1}{2}\left(1 - \frac{\omega_0-\omega}{\Omega}\right)$, yielding the solution

$$a(t) = e^{i\omega t/2}\left(\cos\frac{\Omega t}{2} + i\frac{\omega_0-\omega}{\Omega}\sin\frac{\Omega t}{2}\right)$$

Note that $b(t) = Ce^{-i\omega t/2}\sin\frac{\Omega t}{2}$, however, we do not have another condition to determine the coefficient. So, we will look at $a(t)$ knowing that $|a|^2 + |b|^2 = 1$.

The probability of finding the particle in the spin-up configuration after time t is

$$\begin{aligned}P_{+z\to+z} = |a|^2 &= \cos^2\frac{\Omega t}{2} + \frac{(\omega-\omega_0)^2}{\Omega^2}\sin^2\frac{\Omega t}{2}\\ &= 1 - \left(1 - \frac{(\omega-\omega_0)^2}{\Omega^2}\right)\sin^2\frac{\Omega t}{2}\\ &= 1 - \frac{\omega_1^2}{\Omega^2}\sin^2\frac{\Omega t}{2}\end{aligned}$$

Hence the probability of finding the particle in the spin-down state after time t is

$$P_{+z\to-z} = |b|^2 = \boxed{\frac{\omega_1^2}{\Omega^2}\sin^2\frac{\Omega t}{2}}$$

This suggests that $b(t) = \frac{\omega_1}{\Omega}e^{-i\omega t/2}\sin\frac{\Omega t}{2}$.

Alternative Solution:

In the Schrodinger picture, our Hamiltonian is

$$H = \frac{\hbar}{2}\begin{pmatrix} -\omega_0 & \omega_1 e^{i\omega t} \\ \omega_1 e^{-i\omega t} & \omega_0 \end{pmatrix}$$

and notice that the magnetic field vector draws out a circle that rotates in the clockwise direction as viewed from above. It would be ideal work in an non-inertial frame that also rotates in the clockwise direction so that the magnetic field vector appears static (time independent).

In the interaction picture, both the operators and the state vectors can exhibit time dependence, however by making a good choice in the mapping between the interaction and Schrodinger picture, we can make the Hamiltonian time independent which reduces the problem to a familiar eigenvalue problem that we love so much!

Let's propose

$$\psi_s = e^{iS_z\omega t/\hbar}\psi_i = e^{i\sigma_z\omega t/2}\psi_i$$

where ψ_s is the Schrodinger wave vector and ψ_i is the interaction wave vector. Substituting this into the Schrodinger equation, we have

$$\begin{aligned} i\hbar\dot{\psi}_s &= H\psi_s \\ i\hbar\left(i\frac{\sigma_z\omega}{2}e^{i\sigma_z\omega t/2}\psi_i + e^{i\sigma_z\omega t/2}\dot{\psi}_i\right) &= He^{i\sigma_z\omega t/2}\psi_i \\ i\dot{\psi}_i &= \left(\frac{\sigma_z\omega}{2} + e^{-i\sigma_z\omega t/2}\frac{H}{\hbar}e^{i\sigma_z\omega t/2}\right)\psi_i \end{aligned}$$

So our new interaction Hamiltonian is $H_{rot}/\hbar = \frac{\sigma_z\omega}{2} + e^{-i\sigma_z\omega t/2}He^{i\sigma_z\omega t/2}$. Now, in the S_z basis, H_{rot} can be expressed as

$$\begin{aligned} H_{rot}/\hbar &= \frac{1}{2}\begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} + \frac{1}{2}\begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix}\begin{pmatrix} -\omega_0 & \omega_1 e^{i\omega t} \\ \omega_1 e^{-i\omega t} & \omega_0 \end{pmatrix}\begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -\omega_0 & \omega_1 \\ \omega_1 & \omega_0 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \omega - \omega_0 & \omega_1 \\ \omega_1 & -(\omega - \omega_0) \end{pmatrix} \end{aligned}$$

There is no longer any time dependence in $H_{rot}/\hbar$ and so, we can reduce the problem down to a time-independent Schrodinger equation, or eigenvalue problem

$$\frac{1}{\hbar}H_{rot}\psi_i = \lambda\psi_i$$

Diagonalizing the interaction Hamiltonian, we find that the eigenvalues are

$$\lambda = \pm\frac{1}{2}\sqrt{\omega_1^2 + (\omega - \omega_0)^2} = \pm\frac{\Omega}{2}$$

and the corresponding eigenstates are

$$|\Omega_\pm\rangle = \frac{1}{\sqrt{2\Omega(\Omega \pm (\omega - \omega_0))}}\begin{pmatrix} \Omega \pm (\omega - \omega_0) \\ \pm\omega_1 \end{pmatrix}$$

At time $t = 0$, our original state represented in the S_z basis was

$$|\psi_s(0)\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad |\psi_i(0)\rangle = e^{-i\sigma_z\omega(0)/2}|\psi_s(0)\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and expressed in the basis of H_{rot}, we have

$$\begin{aligned} |\psi_i(0)\rangle &\rightarrow \langle\Omega_+|\psi_i(0)\rangle|\Omega_+\rangle + \langle\Omega_-|\psi_i(0)\rangle|\Omega_-\rangle \\ &= \sqrt{\frac{\Omega + (\omega - \omega_0)}{2\Omega}}|\Omega_+\rangle + \sqrt{\frac{\Omega - (\omega - \omega_0)}{2\Omega}}|\Omega_-\rangle \end{aligned}$$

Evolving this over a time t, our state becomes

$$\begin{aligned} U|\psi_i(0)\rangle &= e^{-iH_{rot}t/\hbar}|\psi(0)\rangle \\ &\rightarrow \sqrt{\frac{\Omega + (\omega - \omega_0)}{2\Omega}}e^{-i\Omega t/2}|\Omega_+\rangle + \sqrt{\frac{\Omega - (\omega - \omega_0)}{2\Omega}}e^{i\Omega t/2}|\Omega_-\rangle \end{aligned}$$

and so, the probability amplitude of finding the state of the particle in the spin-up configuration is

$$\begin{aligned} \langle\uparrow|U|\psi_i(0)\rangle &= \frac{1}{2}\left(1 + \frac{\omega - \omega_0}{\Omega}\right)e^{-i\Omega t/2} + \frac{1}{2}\left(1 - \frac{\omega - \omega_0}{\Omega}\right)e^{i\Omega t/2} \\ &= \cos\frac{\Omega t}{2} - i\frac{\omega - \omega_0}{\Omega}\sin\frac{\Omega t}{2} \end{aligned}$$

Then the probability of finding the particle in spin-up after time t is

$$\begin{aligned} P_{+z\rightarrow+z} &= \cos^2\frac{\Omega t}{2} + \frac{(\omega - \omega_0)^2}{\Omega^2}\sin^2\frac{\Omega t}{2} \\ &= 1 - \left(\frac{(\omega - \omega_0)^2}{\Omega^2} - 1\right)\sin^2\frac{\Omega t}{2} \\ &= 1 - \frac{\omega_1^2}{\Omega^2}\sin^2\frac{\Omega t}{2} \end{aligned}$$

and the probability for spin-down after time t is

$$\boxed{P_{+z\rightarrow-z} = 1 - P_{+z\rightarrow+z} = \frac{\omega_1^2}{\Omega^2}\sin^2\frac{\Omega t}{2}}$$

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