1 J06M3

1.1 (a)

$$\mathcal{L} = \frac{1}{2}m\rho^2\omega^2 + \frac{1}{2}m\dot{l}^2 - mgz$$

where ρ is the cylindrical coordinate that measures the perpendicular distance from the z axis, and l is the coordinate that measures distance from the origin along the wire. Plugging in $z = a(\frac{\rho}{a})^\alpha$, we can get rid of the variable z. The differential length along the wire is:

$$dl = \sqrt{dz^2 + d\rho^2} = d\rho\sqrt{1 + (\frac{dz}{d\rho})^2}$$

Divide this by dt and plug into the Lagrangian as $\dot{l}$. The Lagrangian becomes dependent on only 1 coordinate, namely ρ. Apply the Euler-Lagrange equations to obtain:

$$\frac{d}{dt}\{m\dot{\rho}[1 + \alpha^2 a^{2-2\alpha}\rho^{2\alpha-2}]\} = m\rho\omega^2 + \dot{\rho}^2\frac{\alpha^2(\alpha-1)}{a}(\frac{\rho}{a})^{2\alpha-3}m - mg\alpha(\frac{\rho}{a})^{\alpha-1}$$

Compute the time derivative, cancel the mass, and cancel one of the terms to obtain the equation of motion:

$$\ddot{\rho}[1 + \alpha^2(\frac{\rho}{a})^{2\alpha-2}] + \dot{\rho}^2\frac{\alpha^2(\alpha-1)}{a}(\frac{\rho}{a})^{2\alpha-3} - \rho\omega^2 + g\alpha(\frac{\rho}{a})^{\alpha-1} = 0 \quad (1)$$

1.2 (b)

If the bead is not moving, then its time derivatives are 0 and can be cancelled from the equation of motion:

$$-\rho\omega^2 + g\alpha(\frac{\rho}{a})^{\alpha-1} = 0$$

<u>If $\alpha = 1$</u>:

$$\rho_0 = \frac{g}{\omega^2} \quad (2)$$

<u>If $\alpha = 2$</u>, in the case $\frac{2g}{a} = \omega^2$, all points are equlibrium points; in the case this equality doesn't hold, then the equlibrium point is:

$$\rho_0 = 0 \quad (3)$$

<u>If $\alpha > 2$</u>, then there are two equilibrium points:

$$\rho_0 = \begin{cases} \left(\frac{\omega^2 a^{\alpha-1}}{\alpha g}\right)^{\frac{1}{\alpha-2}} \\ 0 \end{cases} \quad (4)$$

1.3 (c)

If $\alpha = 1$, then we can plug into the equation of motion a solution $\rho = \rho_0 + \epsilon$, to obtain:

$$2\ddot{\epsilon} - \epsilon\omega^2 = 0$$

This corresponds to exponential solutions, so there are no nearby solutions that undergo harmonic motion.

If $\alpha = 2$, in the case $\frac{2g}{a} = \omega^2$, we plug into the equation of motion and find, after linearizing:

$$\ddot{\epsilon}\left[1 + \frac{4}{a^2}\rho_0^2\right] = 0$$

This says that $\ddot{\epsilon} = 0$, which does not correspond to harmonic motion. In the case that $\frac{2g}{a} \neq \omega^2$, the equations of motion say:

$$\ddot{\epsilon} = -\epsilon\left(\frac{2g}{a} - \omega^2\right)$$

If $\frac{2g}{a} - \omega^2 < 0$, this is not a harmonic solution, but if $\frac{2g}{a} - \omega^2 > 0$, this IS a harmonic solution with frequency:

$$\omega_2 = \sqrt{\frac{2g}{a} - \omega^2} \tag{5}$$

If $\alpha > 2$, we plug in $\rho_0 + \epsilon$ as the solution (for the case $\rho_0 \neq 0$) and obtain:

$$\ddot{\epsilon} = -\epsilon\frac{\omega^2(\alpha - 2)}{1 + \alpha^2\left(\frac{\omega^2 a}{\alpha g}\right)^{\frac{2\alpha-2}{\alpha-2}}}$$

And since $\alpha > 2$, this is harmonic motion with frequency:

$$\omega_{>2} = \omega\sqrt{\frac{\alpha - 2}{1 + \alpha^2\left(\frac{\omega^2 a}{\alpha g}\right)^{\frac{2\alpha-2}{\alpha-2}}}} \tag{6}$$

We can now plug in the other solution ($\rho = 0$) into the equation of motion, and get:

$$\ddot{\epsilon} = \epsilon\omega^2$$

This does not correspond to harmonic motion.

1.4 (d)

If $\alpha = 1$, we saw above that we get exponential solutions. To see the long-time behaviour, approximate the equation of motion to the case $\rho >> a$ and $\rho\omega^2 >> g$. We get:

$$2\ddot{\rho} - \rho\omega^2 = 0$$

This corresponds to exponential motion.

If $\alpha = 2$, in the case $\frac{2g}{a} = \omega^2$, we saw that the motion was quadratic and the solution would go away from the z axis. At long times, we would see square root motion (approximate the equation of motion and solve for ρ). In the case that $\frac{2g}{a} < \omega^2$, the initial solution was exponential, and at long times we would find the same as I just described for the case of equality.

If $\alpha > 2$, we saw that the solution $\rho = 0$ was unstable. At long times, we would again see the same behaviour as I just described for the case $\alpha = 2$.