1 January 2006, Mechanics, Problem 2

1.1 (a)

$$U = U(r)$$
$$\boldsymbol{F} = -\frac{dU}{dr}\hat{r}$$
$$\frac{1}{2}mv_A^2 + U(2R) = \frac{1}{2}mv(r)^2 + U(r) \quad E - cons.$$
$$2Rv_A = rvsin(90-\theta) = rvcos\theta \quad L - cons.$$

where θ is the angle between the radius vector and the x axis. If ϕ is the angle between the line that joins the center of the circle with the particle and the x axis, then:

$$\phi = 2\theta$$
$$v(r) = R\dot{\phi}$$
$$r = \sqrt{(Rsin\phi)^2 + (R + Rcos\phi)^2}$$
$$= \sqrt{R^2sin^2\phi + R^2 + R^2cos^2\phi + 2R^2cos\phi}$$
$$= \sqrt{2R^2 + 2R^2cos\phi} = \sqrt{2}R\sqrt{1+cos\phi}$$
$$\frac{1}{2}mv_A^2 + U(2R) = \frac{1}{2}mR^2\dot{\phi}^2 + U(\sqrt{2}R\sqrt{1+cos\phi}$$

We'll guess that the force is of the form $U(r) = \frac{k}{r^n}$.

$$2Rv_A = \sqrt{2}R\sqrt{1+cos\phi}R\dot{\phi}cos(\phi/2)$$
$$2v_A = R\sqrt{1+cos\phi}\dot{\phi}\sqrt{1+cos\phi}$$
$$\frac{1}{2}mv_A^2 + \frac{k}{(2R)^n} = \frac{1}{2}mR^2\dot{\phi}^2 + \frac{k}{(\sqrt{2}R\sqrt{1+cos\phi})^n}$$
$$2v_A = R(1+cos\phi)\dot{\phi}$$
$$\dot{\phi} = \frac{2v_A}{R(1+cos\phi)}$$
$$\frac{1}{2}mv_A^2 + \frac{k}{(2R)^n} = \frac{1}{2}mR^2\frac{4v_A^2}{R^2(1+cos\phi)^2} + \frac{k}{(\sqrt{2}R\sqrt{1+cos\phi})^n}$$
$$2 = n/2$$
$$n = 4$$
$$2mv_A^2 + \frac{k}{4R^4} = 0$$
$$k = -8mR^4v_A^2$$
$$U = -8mR^4v_A^2/r^4$$
$$\boldsymbol{F} = -\frac{32mR^4v_A^2}{r^5}\hat{r} \qquad (1)$$

1.2 (b)

This was already proven in the previous part:

$$E = 0 \tag{2}$$

$$L = 2mRv_A \tag{3}$$

1.3 (c)

$$T = \int_0^T dt = \int_0^{2\pi} \frac{dt}{d\phi}\, d\phi = \int_0^{2\pi} \frac{R(1 + cos\phi)}{2v_A}\, d\phi = \frac{R}{2v_A}[\phi + sin\phi]_0^{2\pi} = \frac{\pi R}{v_A} \tag{4}$$