1 January 2006, Mechanics, Problem 2

1.1 (a) 

\[ U = U(r) \]

\[ F = -\frac{dU}{dr} \hat{r} \]

\[ \frac{1}{2}mv_A^2 + U(2R) = \frac{1}{2}mv(r)^2 + U(r) \quad E - \text{cons.} \]

\[ 2rv_A = rv\sin(90 - \theta) = rv\cos \theta \quad L - \text{cons.} \]

where \( \theta \) is the angle between the radius vector and the x axis. If \( \phi \) is the angle between the line that joins the center of the circle with the particle and the x axis, then:

\[ \phi = 2\theta \]

\[ v(r) = R\dot{\phi} \]

\[ r = \sqrt{(R\sin \phi)^2 + (R + R\cos \phi)^2} \]

\[ = \sqrt{R^2\sin^2 \phi + R^2 + R^2\cos^2 \phi + 2R^2\cos \phi} \]

\[ = \sqrt{2R^2 + 2R^2\cos \phi} = \sqrt{2R\sqrt{1 + \cos \phi}} \]

\[ \frac{1}{2}mv_A^2 + U(2R) = \frac{1}{2}mR^2\dot{\phi}^2 + U(\sqrt{2R}\sqrt{1 + \cos \phi}) \]

We’ll guess that the force is of the form \( U(r) = \frac{k}{r^n} \).

\[ 2rv_A = \sqrt{2R\sqrt{1 + \cos \phi}}R\dot{\phi}\cos(\phi/2) \]

\[ 2v_A = R\sqrt{1 + \cos \phi}\dot{\phi}\sqrt{1 + \cos \phi} \]

\[ \frac{1}{2}mv_A^2 + \frac{k}{(2R)^n} = \frac{1}{2}mR^2\dot{\phi}^2 + \frac{k}{(\sqrt{2R}\sqrt{1 + \cos \phi})^n} \]

\[ 2v_A = R(1 + \cos \phi)\dot{\phi} \]

\[ \dot{\phi} = \frac{2v_A}{R(1 + \cos \phi)} \]

\[ \frac{1}{2}mv_A^2 + \frac{k}{(2R)^n} = \frac{1}{2}mR^2\frac{4v_A^2}{R^2(1 + \cos \phi)^2} + \frac{k}{(\sqrt{2R}\sqrt{1 + \cos \phi})^n} \]

\[ 2 = n/2 \]

\[ n = 4 \]

\[ 2mv_A^2 + \frac{k}{4R^4} = 0 \]

\[ k = -8mR^4v_A^2 \]

\[ U = -8mR^4v_A^2/r^4 \]

\[ F = -\frac{32mR^4v_A^2}{r^5} \hat{r} \quad (1) \]
1.2  (b)

This was already proven in the previous part:

\[ E = 0 \]  (2)

\[ L = 2mRv_A \]  (3)

1.3  (c)

\[ T = \int_0^T dt = \int_0^{2\pi} \frac{dt}{d\phi} d\phi = \int_0^{2\pi} \frac{R(1 + \cos\phi)}{2v_A} d\phi = \frac{R}{2v_A} [\phi + \sin\phi]_0^{2\pi} = \frac{\pi R}{v_A} \]  (4)