1 Mechanics

1.1 Problem 1

(a) There are three angles, and therefore three angular velocities. $\dot{\alpha}$ is in the $\hat{i}_3$ direction, and $\dot{\theta}$ is in the $\hat{i}_1$ direction. $\dot{\theta}$ is in the $\hat{j}$ direction, so in our basis we can write it as a component in the $\hat{i}_2$ direction and a component in the $\hat{i}_3$ direction:

$$\omega_1 = \dot{\theta}$$
$$\omega_2 = \dot{\phi} cos\theta$$
$$\omega_3 = -\dot{\phi} sin\theta + \dot{\alpha}$$

The lagrangian is then:

$$\mathcal{L} = \frac{I\dot{\theta}^2}{2} + \frac{I\dot{\phi}^2 cos^2\theta}{2} + \frac{I_3(\dot{\phi} sin\theta - \dot{\alpha})^2}{2} + MgRsin\theta \quad (1)$$

$$E = \frac{I\dot{\theta}^2}{2} + \frac{I\dot{\phi}^2 cos^2\theta}{2} + \frac{I_3(\dot{\phi} sin\theta - \dot{\alpha})^2}{2} - MgRsin\theta \quad (2)$$

(b) The energy was already written at the end of the previous part. The conserved angular momentum is derived from the lagrangian:

$$\frac{\partial \mathcal{L}}{\partial \dot{\alpha}} = L_\alpha = -I_3(\dot{\phi} sin\theta - \dot{\alpha}) \quad (3)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = L_\phi = I\dot{\phi} cos^2\theta + I_3(\dot{\phi} sin\theta - \dot{\alpha}) sin\theta \quad (4)$$

(c) Plugging L_α into L_ϕ, and solving for $\dot{\phi}$:

$$\dot{\phi} = \frac{L_\phi + L_\alpha sin\theta}{Icos^2\theta}$$

Plugging this and L_α into the energy equation:

$$I\dot{\theta}^2 + \frac{(L_\phi + L_\alpha sin\theta)^2}{Icos^2\theta} + \frac{L_\alpha^2}{I_3} - 2MgRsin\theta - 2E = 0 \quad (5)$$

(d) With the given initial conditions, the conserved quantities are:

$$L_\alpha = L_0$$
$$L_\phi = 0$$
$$E = \frac{L_0^2}{2I_3}$$

Plug into the equation above and make the approximation $\theta << 1$ up to second order, and you get:

$$I\dot{\theta}^2 + \frac{L_0^2\theta^2}{I} - 2MgR\theta = 0$$

Plug in the proposed solution and obtain three equations, one from terms proportional to $cos^2(\omega_n t)$, one from terms proportional to $cos(\omega_n t)$ and one from terms without t-dependence. One of them turns out to be a linear combination of the other two, which are:

$$I\theta_0\omega_n^2 + \frac{L_0^2\theta_0}{I} - 2MgR = 0$$

$$-I\theta_0\omega_n^2 + \frac{L_0^2\theta_0}{I} = 0$$

Adding them and solving for θ_0:

$$\theta_0 = \frac{MgRI}{L_0^2} \tag{6}$$

Then we plug this into the second equation and solve for the frequency:

$$\omega_n = \frac{L_0}{I} \tag{7}$$

Now plug in the conserved quantities, the two constants and the small-θ approximation into the equation for $\dot{\phi}$:

$$\dot{\phi} = \frac{MgR}{L_0}[1 - cos(L_0 t/I)]$$

Since the time average of a cosine is 0:

$$< \dot{\phi} >= MgR/L_0 \tag{8}$$

The condition for θ to be small is that the maximum value of it is much smaller than 1. The maximum value is attained when the cosine is equal to -1, i.e.,

$$\frac{2MgRI}{L_0^2} << 1$$

$$L_0 >> \sqrt{2MgRI} \tag{9}$$