1/2006 CM #1

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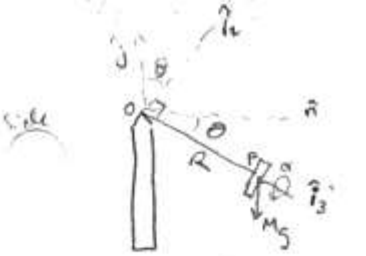


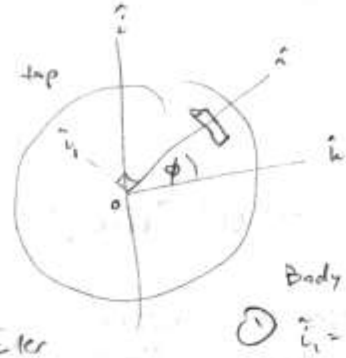


$T = \frac{1}{2}\bar{\omega}\cdot\underline{I}\,\bar{\omega}$ $\qquad V = -Mg\ell\sin\theta$

$I = \begin{pmatrix} I & & \\ & I & \\ & & I_3 \end{pmatrix}$

Body frame: $\bar{\omega} = \omega_1\hat{i}_1 + \omega_2\hat{i}_2 + \omega_3\hat{i}_3$

Key
① - writing body coords in terms of space coords
② solving for rotations in $\alpha, \theta, \phi$ along defined axes in body coords

$\psi\ \theta\ \phi$
$\alpha, \theta, \phi$ are just the Euler angles

Symmetric top
normal $(\theta + \frac{\pi}{2}) = \theta$ for this problem

① $\hat{i}_1 = \cos\phi\,\hat{i} - \sin\phi\,\hat{k}$
$\hat{i}_2 = \cos\theta\,\hat{j} + \sin\theta\cos\phi\,\hat{k} + \sin\theta\sin\phi\,\hat{i}$
$\hat{i}_3 = -\sin\theta\,\hat{j} + \cos\theta\cos\phi\,\hat{k} + \cos\theta\sin\phi\,\hat{i}$

$\Rightarrow T = \frac{1}{2}I(\omega_1^2 + \omega_2^2) + I_3\frac{\omega_3^2}{2}$

②
$\dot{\phi}\hat{j} = A\,i_2 + B\,i_3$
$= (A\cos\theta - B\sin\theta)\hat{j} + (A\sin\theta\cos\phi + B\cos\theta\cos\phi)\hat{k} + (A\sin\theta\sin\phi + B\cos\theta\sin\phi)\hat{i}$ $\Rightarrow A\sin\theta = -B\cos\theta$
$= 0$
$= A\left[\cos\theta + \frac{\sin^2\theta}{\cos\theta}\right] = A\left[\frac{\cos^2\theta + \sin^2\theta}{\cos\theta}\right] = \frac{A}{\cos\theta} \rightarrow$ $A = \dot{\phi}\cos\theta$, $B = -\dot{\phi}\sin\theta$

$T = \frac{I_1}{2}(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2$

Rotation in $\theta$ is in $\hat{i}_1$, Rotation in $\alpha$ is in $\hat{i}_3$

$\Rightarrow \omega_1 = \dot{\theta}$
$\omega_2 = \dot{\phi}\cos\theta$ $\quad \Rightarrow T = \frac{1}{2}I(\dot{\theta}^2 + \dot{\phi}^2\cos^2\theta) + \frac{1}{2}I_3(\dot{\alpha} - \dot{\phi}\sin\theta)^2$
$\omega_3 = -\dot{\phi}\sin\theta + \dot{\alpha}$ $\quad L = T - V = \frac{1}{2}I(\dot{\theta}^2 + \dot{\phi}^2\cos^2\theta) + \frac{1}{2}I_3(\dot{\alpha} - \dot{\phi}\sin\theta)^2 + MgR\sin\theta$

since KE quadratic in $\dot{\theta}, \dot{\alpha}, \dot{\phi}$
$H = E = T + V$

B) $E = \frac{1}{2}I(\dot{\theta}^2 + \dot{\phi}^2\cos^2\theta) + \frac{1}{2}I_3(\dot{\alpha} - \dot{\phi}\sin\theta)^2 - MgR\sin\theta$ = const

only $\dot{\phi}$, $\dot{\alpha}$ appear in Lagrangian $\Rightarrow P_\phi$, $P_\alpha$ are conserved

$P_\phi \equiv \frac{\partial L}{\partial\dot{\phi}} = I\dot{\phi}\cos^2\theta - I_3(\dot{\alpha} - \dot{\phi}\sin\theta)\sin\theta$ const $\quad \searrow \quad P_\phi = I\dot{\phi}\cos^2\theta - \sin\theta\,P_\alpha$
$\dot{\phi}\cos\theta = \frac{1}{I}\left[\frac{P_\phi - \sin\theta P_\alpha}{\cos\theta}\right]$

$P_\alpha \equiv \frac{\partial L}{\partial\dot{\alpha}} = I_3(\dot{\alpha} - \dot{\phi}\sin\theta)$ const

D.) $\Rightarrow 0 = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2I}\left(\frac{P_\phi + \sin\theta P_\alpha}{\cos\theta}\right)^2 + \frac{1}{2I_3}P_\alpha^2 - MgR\sin\theta - E = F(\theta, \dot{\theta})$

For $\theta \ll 1 \Rightarrow$ Approx soln $\theta = \theta_0(1 - \cos\omega_n t)$ $\quad P_\alpha = L_0$, $P_\phi = 0$, $E = \frac{L_0^2}{2I_3}$

E) $t = 0$, $\theta = 0$, $\dot{\phi} = \dot{\theta} = 0$, $\dot{\alpha} = \frac{L_0}{I_3}$

Dropping terms $O(\theta^2)$ and above: $\frac{1}{2}I\dot{\theta}^2 + \underbrace{\frac{1}{2I}(P_\phi + \theta P_\alpha)^2 + \frac{1}{2I_3}P_\alpha^2}_{A} - MgR\theta = E$

$A \longrightarrow \frac{L_0^2}{2I}(\theta - 1)^2 + \frac{L_0^2}{2I_3}$

$\rightarrow \frac{1}{2}I\dot{\theta}^2 = E - A + MgR\theta$

Try soln $\theta = \theta_0(1 - \cos\omega_n t)$
$\dot{\theta} = \theta_0\omega_n\sin\omega_n t$

$\rightarrow \frac{1}{2}I\dot{\theta}^2 + \frac{L_0^2}{2I}\theta^2 + \frac{P_\alpha^2}{2I_3} - MgR\theta = \frac{P_\alpha^2}{2I_3}$

$I\theta_0\omega_n^2\sin^2\omega_n t + \frac{L_0^2}{I}\theta_0(1 + \cos^2\omega_n t - 2\cos\omega_n t) - 2MgR(1 - \cos\omega_n t) = 0$
($\cos^2\omega_n t = 1 - \sin^2\omega_n t$)

$I\theta_0\omega_n^2\sin^2\omega_n t - \sin^2\omega_n t\,\frac{L_0^2}{I}\theta_0 + \frac{2L_0^2}{I}\theta_0(1 - \cos\omega_n t) - 2M_gR(1 - \cos\omega_n t) = 0$

$\Rightarrow \frac{2L_0^2}{I}\theta_0 = 2MgR$

$\dot{\phi} = \frac{1}{I}\frac{(P_\phi - \sin\theta P_\alpha)}{\cos^2\theta}$ ($P_\phi = 0$)
$= -\frac{\theta L_0}{I} = -\frac{\theta_0 L_0}{I}(1 - \cos\omega_n t)$

$\langle\dot{\phi}\rangle = -\frac{\theta_0 L_0}{I} = -\frac{2MgR\,L_0}{L_0^2\,2}$

$\Rightarrow I\theta_0\omega_n^2 = \frac{L_0^2}{I}\theta_0$

$\rightarrow \boxed{\langle\dot{\phi}\rangle = \frac{-MgR}{L_0}}$ $\qquad \boxed{\omega_n^2 = \frac{L_0^2}{I^2}}$ $\qquad \boxed{\theta_0 = \frac{IMgR}{L_0^2}}$

For $\theta \ll 1$ $\forall$ time $\quad \boxed{L_0^2 \gg IMgR}$