Let us consider a Harmonic Oscillator in 3D

$$U = \frac{1}{2}k(x^2 + y^2 + z^2) = \frac{1}{2}m\omega^2 r^2$$

Suppose a particle of mass m and charge q moves in an approximately circular trajectory. This is 2D dynamics in a 3D system. The particle must experience centripetal force

$$F_c = -m\omega^2 r$$

and spring force

$$F_s = -kr$$

Therefore $\omega^2 = k/m$. The 2D circular motion behaves just like 1D linear oscillation.

1 What is the characteristic time?

We wish to compare electromagnetic radiation to kinetic energy. The Lamour formula for power radiation is (in cgs units)

$$P = \frac{2\ddot{d}^2}{3c^3} = \frac{2q^2}{3c^3}a^2 = \frac{2q^2 r^2}{3c^3}\omega^4$$

The kinetic energy of circular orbit is

$$E = \frac{1}{2}m\omega^2 r^2$$

Therefore the characteristic time is

$$\tau = \frac{E}{P} = \frac{3mc^3}{4q^2\omega^2} = \left(\frac{3m^2c^3}{4q^2}\right)\frac{1}{k}$$

2 Adiabatic Condition

What condition must be satisfied for energy radiated per orbit period to be small? The orbit time $T = 2\pi/\omega$ must be small compared to the radiation time τ. Another way to express this is

$$\omega\tau \gg 1$$

Therefore we need

$$\omega < \frac{3mc^3}{4q^2}$$

An upper bound on ω makes sense, because power radiated $P \to \infty$ as $\omega \to \infty$.

3 Radiation Force

The Abraham Lorentz radiation reaction force is given by

$$F_r = -\frac{2q^2}{3c^3}\dot{a}$$

notice it makes sense that $P = \dot{F}$. But for circular motion $\dot{a} = \omega^3 r$. Therefore the ratio of forces is

$$\frac{F_r}{F_s} = \frac{2q^2\omega^3 r/3c^3}{m\omega^2 r} = \frac{2q^2\omega}{3c^3 m} \approx (\omega\tau)^{-1} \ll 1$$

where the final inequality follows from our adiabatic condition. [1]

[1] For SI units, the power is given by

$$P' = \frac{P}{4\pi\epsilon_0} = \frac{q^2 a^2}{6\pi\epsilon_0 c^3}$$

So the time constant is

$$\tau' = \frac{E}{P'} = 4\pi\epsilon_0\tau = \frac{3\pi\epsilon_0 m^2 c^3}{q^2 k}$$

Similarly the rest of the dimensionless ratios follow.