1 January 2006, Electromagnetism, Problem 3

1.1 (a)

The problem can be solved in two ways:

(i) by realizing that charged particle moving in a circle can be visualized as a superposition of two oscillating dipoles:

$$x(t) = x_0 cos(\omega t)$$
$$y(t) = y_0 sin(\omega t)$$

and using the formulas for dipole radiation; or

(ii) by using Larmor's formula for the radiation of an accelerated charged particle.

(i) Look at Griffiths' "Introduction to Electrodynamics", chapter 9.1, to see a derivation of the formula for the power radiated by an oscillating dipole. In general, in order to find the power radiated by the superposition of two oscillating dipoles, we'd have to sum the values of the fields, compute the Poynting vector, and from there calculate the power. However, in our case, since the dipoles are perpendicular to each other there is no need for this, and we can just add the power due to each dipole. The final answer is:

$$P = \frac{\omega^4 q^2 x_0^2}{6\pi\epsilon_0 c^3}$$

(ii) Look at Griffiths, section 9.2, to see a derivation of the Larmor formula, which gives the power due to any accelerated charged particle:

$$P = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2 a^2}{c^3}$$

The particle is moving in a circle, so the acceleration will be its centripetal acceleration $a = x_0\omega^2$, and plugging this in we see that both equations are the same. For the prelim we'll probably have to know both, but for this particular problem, Larmor's formula proves to be a little more useful (see part c).

Now notice that the power dissipated is equal to the loss of kinetic energy. Meanwhile, the kinetic energy is:

$$T = \frac{1}{2}m\omega^2 x_0^2 = \frac{1}{2}Kx_0^2$$

With this, and the fact that $\omega = \sqrt{K/m}$, we get:

$$\frac{dT}{dt} = -\frac{Kq^2 T}{3\pi\epsilon_0 c^3 m^2}$$

From here we can read off the time constant, noticing that the solution is a decaying exponential:

$$\tau = \frac{3\pi\epsilon_0 c^3 m^2}{Kq^2} \quad (1)$$

1.2 (b)

The condition we need is that the time constant is much larger than the period of the motion:

$$\tau >> \frac{2\pi x_0}{x_0 \omega}$$

$$\frac{2K^{1/2}q^2}{3\epsilon_0 c^3 m^{3/2}} << 1 \tag{2}$$

1.3 (c)

Here there are two ways:

(i) remember the radiation-reaction force formula; or

(ii) "derive it" from Larmor's formula.

Here's where remembering Larmor's formula can prove useful. If the system oscillates one full period and comes back with negligible radius change, then during that time period the power radiated is the same as the loss of energy due to work under a radiation reaction force:

$$\int_{t_1}^{t_2} \boldsymbol{F_{rad}} \boldsymbol{v} \, dt = -\frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{q^2}{c^3} \int_{t_1}^{t_2} a^2 \, dt$$

Integrate the right integral by parts and use the fact that the system has gone through a full period with negligible changes to rule out the boundary terms, and obtain:

$$\int_{t_1}^{t_2} \boldsymbol{F_{rad}} \boldsymbol{v} \, dt = \int_{t_1}^{t_2} \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{q^2}{c^3} \dot{\boldsymbol{a}} \boldsymbol{v} \, dt$$

There is an obvious formula for $\boldsymbol{F_{rad}}$ that would satisfy this equation, and though we haven't proved it, it turns out to be correct:

$$\boldsymbol{F_{rad}} = \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{q^2}{c^3} \dot{\boldsymbol{a}}$$

For an even less convincing argument, go back to this formula:

$$P = -\boldsymbol{F_{rad}} \boldsymbol{v}$$

(which is incorrect to begin with, see Griffiths 9.3 for an explanation), and notice that P is proportional to a^2. So we basically want:

$$\frac{a^2}{\boldsymbol{v}} = \frac{\frac{d\boldsymbol{v}}{dt}\frac{d\boldsymbol{v}}{dt}}{\boldsymbol{v}} = \frac{d^2\boldsymbol{v}}{dt^2}$$

And if you're not convinced by either argument, then you'll have to remember the radiation-reaction force formula (or go through a more complicated derivation, which cannot be found in Griffiths).

Here I have two ways of proving it, and I don't know which is correct

First I will write $\dot{a}$ as $\omega^2 \dot{x_0}$. This assumes that ω is not changing around the circle (seems reasonable because K is fixed). Then:

$$F_{rad} = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2}{c^3}\omega^2\dot{x_0} << \frac{1}{4\pi}\omega\dot{x_0}m = \frac{\omega\dot{T}m}{4\pi K x_0} = \frac{2q^2x_0^2\omega^4}{2*6\pi\epsilon_0 c^3}\frac{1}{4\pi\omega x_0} << \frac{x_0\omega^2}{16\pi^2 m} = \frac{Kx_0}{16\pi^2} < F_{spring}$$

Alternatively, I can write $a = \omega^3 x_0$. The motivation behind this is purely dimensional. Then:

$$F_{rad} = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{q^2}{c^3}\omega^3 x_0 << \frac{\omega^2 x_0}{4\pi m} = \frac{Kx_0}{4\pi} < F_{spring}$$

As you can see, we get the same answer in both cases, but I'd still like to know which one is correct.