1 January 2006, Electromagnetism, Problem 1

1.1 (a)

Want to solve Laplace's equation in cylindrical coordinates, so we need to know the laplacian in cylindrical coordinates. The easiest way to derive it (for me) relies on some knowledge of differential geometry. The formula for the Laplacian is:

$$\nabla^2 V = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{g} g^{\mu\nu} \frac{\partial V}{\partial x^\nu} \right)$$

where $g_{\mu\nu}$ is the metric, so that $g^{\mu\nu}$ is the inverse metric, and g is the determinant of the metric. Of course, we understand summation over repeated indices. We can thus derive the formula:

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial^2 \phi} = 0$$

There is no z dependence because the cylinder is infinitely long, so there must be translation invariance. We assume that the solution is separable as $V(\rho, \phi) = P(\rho)\Phi(\phi)$ and obtain a general solution:

$$V(\rho, \phi) = A ln\rho + B + \sum_{n=1}^{\infty} (C_n \rho^n + D_n \rho^{-n})[\gamma_n sin(n\phi) + \delta_n cos(n\phi)]$$

We assume that the electric field is in the x direction, and so the ϕ dependence must be evenly symmetric. We thus throw out all the sine terms. The physics are invariant under an addition or subtraction of a constant from V, so we will subtract B and avoid having to compute it. We obtain:

$$V_{out}(\rho, \phi) = A ln\rho + \sum_{n=1}^{\infty} (C_n \rho^n + D_n \rho^{-n}) \delta_n cos(n\phi)$$

$$V_{in}(\rho, \phi) = A' ln\rho + \sum_{n=1}^{\infty} (C'_n \rho^n + D'_n \rho^{-n}) \delta'_n cos(n\phi)$$

By the finiteness of the potential at the origin, $A' = D'_n = 0$. Also, for large ρ, the potential must match the potential due to the constant electric field. This potential is:

$$V_E = -E\rho cos(\phi) + f(\phi)$$

where f is some function of ϕ. f must satisfy Laplace's equation as well, which implies that f must be a linear function of ϕ. But then it would not be an even function of ϕ unless it were a constant, in which case we can just throw it out. Now we match this potential with the one we have at large ρ:

$$Aln\rho + \sum_{n=1}^{\infty}(C_n\rho^n)\delta_n cos(n\phi) = -E\rho cos(\phi)$$

This sets $A = \delta_{n\neq 1} = 0$, $C_1\delta_1 = -E$, which gives us our final potential (after absorbing δ_1 into D_1 and δ'_n into C'_n):

$$V_{out} = \left(-E\rho + \frac{D_1}{\rho}\right)cos(\phi)$$

$$V_{in} = \sum_{n=1}^{\infty} C'_n\rho^n cos(n\phi)$$

You can match the two potentials at the boundary to determine:

$$C'_n = 0 \quad n \neq 1$$

$$\frac{D_1}{a} - Ea = C'_1 a$$

One more condition will complete the potential. That is the continuity of the displacement vector's perpendicular component across the boundary (this is because there is no free charge). To get that, you need to know the ρ component of the gradient of a scalar. The way I remember this is that the gradient is made up of the "normalized forms". This means, in a nutshell, that if the derivative with respect to x, y, z, r or ρ, it's just the partial derivative with respect to that variable, and if it is a derivative with respect to θ, ϕ_{polar} or $\phi_{cylindrical}$, it is $\frac{1}{\sqrt{g^{kk}}}$ times the derivative with respect to the variable, where g^{kk} means the diagonal component of the metric that corresponds to the variable under consideration. Anyways, we want to write this boundary condition:

$$\epsilon\frac{\partial V_{in}}{\partial\rho} = \frac{\partial V_{out}}{\partial\rho}$$

$$V_{in} = -\frac{2E}{1+\epsilon}\rho cos\phi \quad \rho < a \tag{1}$$

$$V_{out} = -E\rho cos\phi + \frac{2E(\epsilon - 1)}{\epsilon + 1}\frac{cos\phi}{\rho} \quad \rho > a \tag{2}$$

1.2 (b)

Compute the gradient of V_{in} and then use:

$$\vec{E} = -\nabla V = \frac{2E}{1+\epsilon}(cos\phi\hat{\rho} - sin\phi\hat{\phi}) \tag{3}$$

$$\vec{D} = \epsilon\epsilon_0\vec{E} = 2\epsilon_0 E\frac{\epsilon}{1+\epsilon}(cos\phi\hat{\rho} - sin\phi\hat{\phi}) \tag{4}$$

1.3 (c)

Somehow I remember that the bound charges are computed from the polarization vector, but I can never remember what the polarization vector is. Tough luck, I'll just have to:

$$\vec{P} = \vec{D} - \epsilon_0 \vec{E}$$

If you know Gauss' law and its analogous formula for the displacement vector, you'll find:

$$\nabla \cdot \vec{P} = -\rho_B$$

The other one is not so obvious, but I kind of remember that one of the bound charges is $\vec{P}\hat{n}$, and I already saw that it's not the volume charge, so it must be the surface charge:

$$\sigma_B = \vec{P} \cdot \hat{n} = 2\epsilon_0 E \left(\frac{\epsilon - 1}{\epsilon + 1} \right) cos\phi \tag{5}$$

Now you have to remember the divergence in spherical coordinates. This is just too bad, because I know no formula or trick from which I can derive it, so I have to remember it. Look in Griffiths [**?**] for a formula. It turns out it has two terms that cancel each other:

$$\rho_B = 0 \tag{6}$$

1.4 (d)

Here we know the formula for free space:

$$u_{vacuum} = \frac{\epsilon_0}{2} \int_{all\, space} E^2 \, d\tau$$

We need to modify this to calculate it inside the dielectric. Griffiths motivates a derivation of the actual formula, but it's a little long to reproduce here. A bullshit argument (more of a mnemonic) is that the original formula contains an E^2 that suggests it is $\vec{E}$ producing work on charges as $\vec{E}$ is established. Here, instead, we have $\vec{E}$ producing work on charges as $\vec{D}$ is established by the free charges. Hence, we write the formula:

$$u = \frac{\epsilon_0}{2} \int_{all\, space} (\vec{D} \cdot \vec{E}) \, d\tau$$

$$\frac{du}{dz} = \frac{\epsilon_0}{2} \int_0^a \int_0^{2\pi} (\vec{D} \cdot \vec{E}) \, \rho d\phi d\rho = \frac{\epsilon\epsilon_0}{2} \left(\frac{2E}{1 + \epsilon} \right)^2 \pi a^2 \tag{7}$$