jost3

a. $S = a(UNV)^{1/3}$

$dU = TdS - PdV + \mu dN \rightarrow \frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_{N,V}$

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_{N,V} = a(NV)^{1/3}\,\frac{1}{3}U^{-2/3}$$

$$\rightarrow T = \frac{3}{a}\left(\frac{U^2}{NV}\right)^{1/3}$$

b. $$C_{VN} = \left(\frac{\partial U}{\partial T}\right)_{N,V} = \frac{\partial}{\partial T}\left(\frac{a(NV)^{1/3}}{3}T\right)^{3/2} = \frac{3}{2}T^{1/2}N^{1/2}V^{1/2}\frac{a^{3/2}}{3^{3/2}}$$

c. $$dU_1 + dU_2 = 0 \rightarrow \int_{T_1}^{T_f} C_1\,dT_1 + \int_{T_2}^{T_f} C_2\,dT_2 = 0$$

$$\int_{T_1}^{T_f} T_1^{1/2}\,dT_1 + \int_{T_2}^{T_f} T_2^{1/2}\,dT_2 = 0$$

$$T_f^{3/2} - T_1^{3/2} + T_f^{3/2} - T_2^{3/2} = 0 \rightarrow T_f = \left(\frac{T_1^{3/2} + T_2^{3/2}}{2}\right)^{2/3}$$

d. $$\eta = \frac{W}{Q_H} = \frac{Q_H - Q_C}{Q_H} \qquad \text{wlog, assume } T_1 > T_2$$

$$1 - \frac{T_2}{T_1} = \frac{-C_1\,dT_1 - C_2\,dT_2}{-C_1\,dT_1} \rightarrow \frac{-T_2}{T_1} = \frac{C_2}{C_1}\frac{dT_2}{dT_1}$$

where from (b) we see $\frac{C_2}{C_1} = \left(\frac{T_2}{T_1}\right)^{1/2}$ hence

$$\frac{-T_2}{T_1} = \frac{T_2^{1/2}}{T_1^{1/2}}\frac{dT_2}{dT_1} \rightarrow \int_{T_1}^{T_f} \frac{-dT_1}{T_1^{1/2}} = \int_{T_2}^{T_f}\frac{dT_2}{T_2^{1/2}}$$

$$-2\left(T_f^{1/2} - T_1^{1/2}\right) = 2\left(T_f^{1/2} - T_2^{1/2}\right)$$

$$T_f' = \left(\frac{T_1^{1/2} + T_2^{1/2}}{2}\right)^{2}$$

$$W = \int_{T_1}^{T_f} -C_1\,dT_1 + \int_{T_2}^{T_f} -C_2\,dT_2 = -\frac{3}{2}\left(\frac{NVa^3}{3^3}\right)^{1/2}\left[\frac{2}{3}\right]\left[T_f'^{3/2} - T_1^{3/2} + T_f'^{3/2} - T_2^{3/2}\right]$$

plugging in T_f' from above.