1 January 2005, Thermodynamics, Problem 2

1.1 (a)

This seems like a very hard problem, then you try it out and it doesn't seem so bad, then you do it and it's as bad as it seemed to be in the beginning.

$$Z = \sum_{n=0}^{\infty} g_n e^{-E_n/kT} = \frac{e^{-3\hbar\omega_0/2kT}}{2}[\sum_{n=0}^{\infty} n^2 e^{-n\hbar\omega_0/kT} + 3\sum_{n=0}^{\infty} ne^{-n\hbar\omega_0/kT} + 2\sum_{n=0}^{\infty} e^{-n\hbar\omega_0/kT}] = \frac{e^{-3\hbar\omega_0/2kT}}{(1-e^{-\hbar\omega_0/kT})^3}$$

$$F = -NkTln(Z) = NkT\left(\frac{3\hbar\omega_0}{2kT} + 3ln(1-e^{-\hbar\omega_0/kT})\right)$$

$$\mu = \left(\frac{\partial F}{\partial N}\right)_T = kT\left(\frac{3\hbar\omega_0}{2kT} + 3ln(1-e^{-\hbar\omega_0/kT})\right)$$

$$U = -NkT\omega_0\frac{\partial(lnZ)}{\partial\omega_0} = \frac{3N\hbar\omega_0}{2} + \frac{3N\hbar\omega_0}{e^{\hbar\omega_0/kT}-1}$$

$$c_N(T) = \left(\frac{\partial U}{\partial T}\right)_N = \frac{3N(\hbar\omega_0)^2 e^{\hbar\omega_0/kT}}{kT^2(1-e^{\hbar\omega_0/kT})^2}$$

Now we have all the quantities, let's make the approximations. In the classical limit:

$$c_N(T) \approx 3Nk + \frac{3N\hbar\omega_0}{T} \tag{1}$$

1.2 (b)

In the quantum limit:

$$c_N(T) \approx 3N\hbar\omega_0 e^{-\hbar\omega_0/kT} \tag{2}$$

1.3 (c)

$$\mu \approx 3kTln\left(\frac{\hbar\omega_0}{kT}\right) \tag{3}$$

The classical limit is attained when this term of the chemical potential is much larger in absolute value than the term of next order:

$$kTln\left(\frac{kT}{\hbar\omega_0}\right) >> \frac{\hbar\omega_0}{2} \tag{4}$$

1.4 (d)

The problem with it is that the hint they give has nothing to do with the way they did it, and I have no expression for N_0 in the whole problem. So with due respect to the problem-writer, I will do it my way. At low temperatures, the ground state is the only one occupied (more or less):

$$N \approx \frac{1}{e^{(\epsilon_0-\mu)/T}-1}$$

where I have used the Bose-Einstein distribution. Since N is very large, so the denominator must be very small, so the exponent must be very close to 1. Expanding:

$$N \approx \frac{1}{1+(\epsilon_0-\mu)/T-1} = \frac{1}{(\epsilon_0-\mu)/T} = \frac{T}{\epsilon_0-\mu}$$

$$\mu = \epsilon_0 - \frac{kT}{N} \tag{5}$$

1.5 (e)

T_{BEC} is defined to be the temperature at which N_0 ceases to be macroscopic. At that point, the chemical potential should be 0, so that it's equally likely for particles to be in the ground state or in the excited state:

$$N = \int_0^N g(n)\frac{1}{e^{(\epsilon_0+n\omega_0)/T_{BEC}}-1}\,dn \approx \int_0^\infty \frac{n^2+3n+2}{2}\frac{1}{e^{n\omega_0/T_{BEC}}-1}\,dn =$$

$$= \int_0^\infty \frac{(T_{BEC}x/\omega_0)^2+3(T_{BEC}x/\omega_0)+2}{2}\frac{1}{e^x-1}\frac{T_{BEC}}{\omega_0}\,dx \approx$$

$$\approx \left(\frac{T_{BEC}}{\omega_0}\right)^3\frac{1}{2}\int_0^\infty \frac{x^2}{e^x-1}\,dx$$

$$I \equiv \int_0^\infty \frac{x^2}{e^x-1}\,dx$$

$$T_{BEC} = \left(\frac{2N}{I}\right)^{1/3}\frac{\hbar\omega_0}{k_B} \tag{6}$$