J05Q.3

J05Q.3 - Heavy Particle Passing a Hydrogen Atom

According to time-dependent perturbation theory, the probability of a particle jumping from state k to k' is,

$$p_{k'k} = \frac{1}{\hbar^2} \left| \int_{-\infty}^{\infty} dt e^{\frac{i}{\hbar}(E_{k'}-E_k)t} < \psi_{k'} | H'(t) | \psi_k > \right|^2 \tag{1}$$

Set the coordinate system with the hydrogen atom as the origin and trajectory of the heavy particle in x-z plane. The coordinate of the heavy particle can be denoted as $(vt, 0, D)$. Thus the distance between the heavy particle and the electron is,

$$s^2(t) = v^2t^2 + D^2 - 2(vtsin\theta)cos\phi + 2Dcos\theta)r + r^2 \tag{2}$$

As $D \gg a$,

$$\frac{1}{s(t)} \simeq \frac{1}{\sqrt{v^2t^2 + D^2}} + (v^2t^2 + D^2)^{-\frac{3}{2}} (vtsin\theta cos\phi + Dcos\theta)r \tag{3}$$

Thus the perturbation Hamiltonian is,

$$H' = -\frac{e^2}{4\pi\varepsilon_0 s(t)} \tag{4}$$

Knowing that,

$$\psi_{10} = \frac{1}{\sqrt{\pi}} a^{-3/2} e^{-r/a}, \tag{5}$$

$$\psi_{210} = \frac{1}{4\sqrt{2\pi}} a^{-3/2} \frac{r}{a} e^{-r/2a} cos\theta, \tag{6}$$

$$\psi_{21,\pm1} = \frac{1}{8\sqrt{\pi}} a^{-3/2} \frac{r}{a} e^{-r/2a} sin\theta e^{\pm i\phi}, \tag{7}$$

We can calculate,

$$\begin{aligned} &< \psi_{210} |H'(t)|\psi_{10} > \\ = & \qquad -\frac{e^2}{4\pi\varepsilon_0} \int_0^\infty r^2 dr \int_0^\pi sin\theta d\theta \\ &\int_0^{2\pi} d\phi (v^2 t^2 + D^2)^{-\frac{3}{2}} D cos\theta r \frac{1}{4\pi\sqrt{2}} a^{-4} r e^{-3r/2a} cos\theta \\ = & -\frac{aDe^2}{12\sqrt{2}\pi\varepsilon_0} (v^2 t^2 + D^2)^{-\frac{3}{2}} \int_0^\infty d(\frac{r}{a})(\frac{r}{a})^4 e^{-\frac{3r}{2a}} \\ = & -\frac{2^6 aDe^2}{3^5\sqrt{2}\pi\varepsilon_0} (v^2 t^2 + D^2)^{-\frac{3}{2}} \end{aligned}$$

$$\begin{aligned} &< \psi_{21,\pm1} |H'(t)|\psi_{10} > \\ = & \qquad -\frac{e^2}{4\pi\varepsilon_0} \int_0^\infty r^2 dr \int_0^\pi sin\theta d\theta \\ &\int_0^{2\pi} d\phi (v^2 t^2 + D^2)^{-\frac{3}{2}} vt sin\theta cos\phi r \frac{1}{8\pi} a^{-4} r e^{-3r/2a} sin\theta e^{\pm i\phi} \\ = & -\frac{2^5 ae^2}{3^5 \pi\varepsilon_0} vt (v^2 t^2 + D^2)^{-\frac{3}{2}} \end{aligned}$$

Then, as $\frac{D|E_0|}{\hbar v} \ll 1$

$$\begin{aligned} &\int_{-\infty}^{\infty} dt e^{\frac{i}{\hbar}\frac{3}{4}E_0 t} H'_{10\to 210} \\ &= -\int_{-\infty}^{\infty} dt e^{\frac{i}{\hbar}\frac{3}{4}E_0 t} \frac{2^6 aDe^2}{3^5\sqrt{2}\pi\varepsilon_0} (v^2 t^2 + D^2)^{-\frac{3}{2}} \\ &\simeq -\int_{-\infty}^{\infty} dt (1 + \frac{i}{\hbar}\frac{3}{4}E_0 t) \frac{2^6 aDe^2}{3^5\sqrt{2}\pi\varepsilon_0} (v^2 t^2 + D^2)^{-\frac{3}{2}} \\ &= -\frac{2^6 ae^2}{3^5\sqrt{2}\varepsilon_0 Dv} \end{aligned}$$

$$\begin{aligned}
&\int_{-\infty}^{\infty} dt e^{\frac{i}{\hbar}\frac{3}{4}E_0 t}\; H'_{10\to 21,\pm 1} \\
&= -\int_{-\infty}^{\infty} dt e^{\frac{i}{\hbar}\frac{3}{4}E_0 t}\; \frac{2^5 a e^2}{3^5 \pi \varepsilon_0}(vt)(v^2t^2+D^2)^{-\frac{3}{2}} \\
&\simeq -\int_{-\infty}^{\infty} dt(1+\frac{i}{\hbar}\frac{3}{4}E_0 t)\, \frac{2^5 a e^2}{3^5 \pi \varepsilon_0}(vt)(v^2t^2+D^2)^{-\frac{3}{2}} \\
&= -\frac{2^4 a e^2 i E_0}{3^5 \pi \varepsilon_0 \hbar v^2}
\end{aligned}$$

Thus, the prbability of the electron be in a $2p$ state is,

$$P_{10\to 210} = \frac{2^{11} a^2 e^4}{\hbar^2 3^{10} \varepsilon_0^2 D^2 v^2} \tag{8}$$

$$P_{10\to 21,\pm 1} = \frac{2^8 a^2 e^4 E_0^2}{3^{10} \pi^2 \varepsilon_0^2 \hbar^4 v^4} \tag{9}$$

One thought on “J05Q.3”

December 15, 2013 at 7:19 pm

All the steps are correct. If there are no computational mistakes, everything's correct so far.
You're just missing one more step -- to find the total probability of $1s \to 2p$ transition.
