Solution to J05Q2

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a) The Hamiltonian is

$$H = |\mathbf{p}|^2/2m - ke^2/R - eER\cos\theta \tag{1}$$

The eigenstate of unperturbed H_0 satisfies periodic boundary condition $\psi^0(\theta + 2\pi m) = \psi^0(\theta)$

$$\psi_n^0(\theta) = 1/\sqrt{2\pi}e^{in\theta} \tag{2}$$

and the energy are:

$$E_n = \frac{n^2\hbar^2}{2mR^2} - \frac{ke^2}{R} \tag{3}$$

First order perturbation:

$$E_0^{(1)} = \langle\psi_0|H_1|\psi_0\rangle \sim \int_0^{2\pi} d\theta\cos\theta = 0 \tag{4}$$

So we need to carry onto the second order:

$$E_0^{(2)} = \sum_{k\neq 0} \frac{|\langle\psi_k|H_1|\psi_0\rangle|^2}{E_0 - E_k} \tag{5}$$

where $\langle\psi_k|H_1|\psi_0\rangle = -\frac{EeR}{2\pi}\int_0^{2\pi} d\theta e^{-ik\theta}(e^{i\theta} + e^{-i\theta})/2$ is only nonzero if $k = \pm 1$. Thus, we have

$$E_0^{(2)} = \sum_{k=\pm 1} \frac{|\langle\psi_k|H_1|\psi_0\rangle|^2}{E_0 - E_k} = \frac{mE^2e^2R^4}{\hbar^2} \tag{6}$$

b) If E is super big, we can think in classical picture that the particle is confined to move at small angle $\theta \sim 0$. Thus we can expand $\cos\theta = 1 - \theta^2/2$, and obtain a harmonic potential:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_n}{\partial\theta^2} + \frac{1}{2}m\omega^2\psi = (E_n + eER + \frac{ke^2}{R})\psi_n \tag{7}$$

where $\omega = \frac{eER}{m}$ and $E_n = \hbar\omega(n + 1/2)$. The wavefunctions are Hermite polynomial.