1 January 2005, Quantum Mechanics, Problem 1

As Hans pointed out to me, if you are asked to get the density matrix, you need the state of the system, because the density matrix is:

$$\rho = |\Psi >< \Psi|$$

So we need to get the state of the system at an arbitrary time. This is relatively easy if you've done similar problems before. First, express the Hamiltonian in a more convenient form:

$$H = -J\left(\frac{S^2 - S_A^2 - S_B^2}{2}\right)$$

where $\boldsymbol{S} = \boldsymbol{S_A} + \boldsymbol{S_B}$. The eigenstates of the system are eigenstates of definite value of total spin. These, can be expressed, in terms of the states of definite value of the z-component of each spin (where the + sign means an "up" state, and the order is always A first, then B):

$$|11 >= | + + >$$
$$|10 >= \frac{1}{\sqrt{2}}(| + - > +| - + >)$$
$$|1-1 >= | - - >$$
$$|00 >= \frac{1}{\sqrt{2}}(| + - > -| - + >)$$

with respective energies:

$$E_{11} = E_{10} = E_{1-1} = -\frac{J}{4}$$
$$E_{00} = \frac{3J}{4}$$

The initial state is $|\Psi(0) >= | + - >= \frac{|10>+|00>}{\sqrt{2}}$, and the state at time t is:

$$|\Psi(t) >= Ae^{-iE_{00}t}|00 > +Be^{-iE_{11}t}|11 > +Ce^{-iE_{10}t}|10 > +De^{-iE_{1-1}t}|1-1 >$$

The initial condition determines the state as:

$$|\Psi(t) >= \frac{1}{\sqrt{2}}[e^{-i3Jt/4}|00 > +e^{iJt/4}|10 >] = e^{-iJt/4}[cos(Jt/2)| + - > +i\,sin(Jt/2)| - + >]$$

The density matrix for the whole system is what I said in the beginning of the problem. But if we want the density matrix for A only, we have to use the formula, from Landau-Lifschitz:

$$\rho_A(x, x') = \int \Psi(q, x)\Psi * (q, x')dq$$

where the interpretation is that x are the coordinates of A and q are the coordinates of B. In matrix notation, this can be written as:

$$\rho_A = \sum_{S_B} < S_B|S_A S_B >< S'_A S_B|S_B >$$

This is kind of hard to write in the language that we have, but it turns out that it is equal to:

$$\rho_A = \sum_{S_B} < S_B|\rho|S_B >$$

When we write this we obtain:

$$\rho_A = cos^2(Jt/2)| < -_B| + - > |^2 + sin^2(Jt/2)| < +_B| - + > |^2$$
$$\rho_A = cos^2(Jt/2)|+_A >< +_A| + sin^2(Jt/2)|-_A >< -_A| \quad (1)$$

In matrix notation:

$$\rho_A = \begin{pmatrix} cos^2(Jt/2) & 0 \\ 0 & sin^2(Jt/2) \end{pmatrix} \quad (2)$$

This will describe a pure state whenever the average value of the z-component of the spin of A is either 1/2 or -1/2.

$$< S_z >= Tr(S_z \rho_A) = \frac{cos^2(Jt/2)}{2} - \frac{sin^2(Jt/2)}{2}$$

Thus, we want the argument of the sine and cosine to be an integer multiple of $\pi/2$:

$$t = \frac{n\pi}{J} \quad (3)$$