1 January 2005, Mechanics, Problem 3

1.1 (a)

$$mgl(1-cos\theta) \approx mgl\theta^2/2 \approx mg\psi_n^2/(2l) = U_g$$

$$l\theta \approx \psi_n$$

$$U_s = \frac{1}{2}K(\psi_{n+1}-\psi_n)^2 + \frac{1}{2}K(\psi_n-\psi_{n-1})^2$$

$$\mathcal{L} = \frac{1}{2}m\dot{\psi}_n^2 - \frac{mg\psi_n^2}{2l} - \frac{1}{2}K(\psi_{n+1}-\psi_n)^2 - \frac{1}{2}K(\psi_n-\psi_{n-1})^2$$

$$m\ddot{\psi}_n = -\frac{mg\psi_n}{l} + K(\psi_{n+1}-\psi_n) - K(\psi_n-\psi_{n-1}) = -\frac{mg\psi_n}{l} + K(\psi_{n+1}-2\psi_n+\psi_{n-1}) \quad (1)$$

1.2 (b)

We want a dispersion relation of the propagation modes:

$$\omega = f(k)$$

I suppose the propagating modes are waves, something like:

$$\psi(x) = Acos(kx-\omega t)$$

but in the discreet case, so that:

$$\psi_n = Acos(kna_0-\omega t)$$

Plug into the equation of motion:

$$\psi_{n+1} = Acos[k(n+1)a_0-\omega t] = A[cos(kna_0-\omega t)cos(ka_0) - sin(kna_0-\omega t)sin(ka_0)]$$

$$\psi_{n-1} = A[cos(kna_0-\omega t)cos(ka_0) + sin(kna_0-\omega t)sin(ka_0)]$$

$$-m\omega^2 Acos(kna_0-\omega t) = -\frac{mg}{l}Acos(kna_0-\omega t) +$$

$$+K\left(A[cos()cos(ka_0) - sin()sin(ka_0)] - 2Acos() + A[cos()cos(ka_0) + sin()sin(ka_0)]\right)$$

$$-m\omega^2 = -\frac{mg}{l} + K[2cos(ka_0)-2]$$

$$m\omega^2 = \frac{mg}{l} + 2K[1-cos(ka_0)] \quad (2)$$

1.3 (c)

What is the range in which they can propagate?

$$m\omega^2 \geq \frac{mg}{l}$$

$$m\omega^2 \leq \frac{mg}{l} + 4K$$

$$\sqrt{\frac{g}{l}} \leq \omega \leq \sqrt{\frac{4K}{m} + \frac{g}{l}} \tag{3}$$