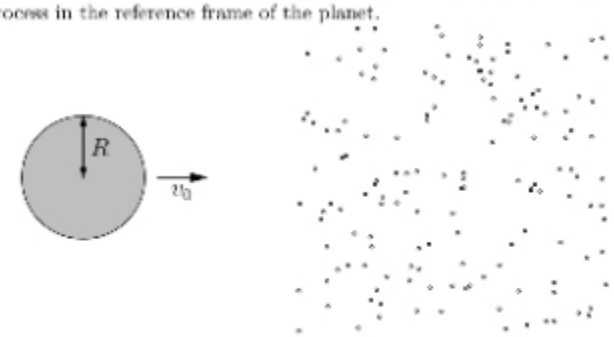
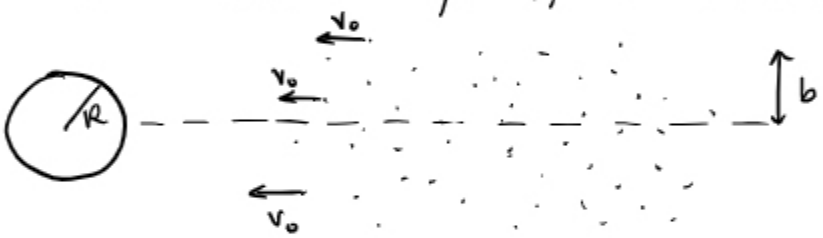
J05M.2 - Planet Moving Through Dust Cloud

Problem

A planet of mass M and radius R moves through a cloud of interplanetary dust at a constant velocity v_0. The dust particles have negligible mass. Depending on its initial position when the planet is still far away, each dust particle will either hit or miss the planet as it moves by. When they strike the planet, they stick. The capture cross section σ is defined as the transverse area within which all dust particles are captured. Compute σ. *Hint:* it is useful to consider the capture process in the reference frame of the planet.



In the reference frame of the planet, dust moves towards the planet

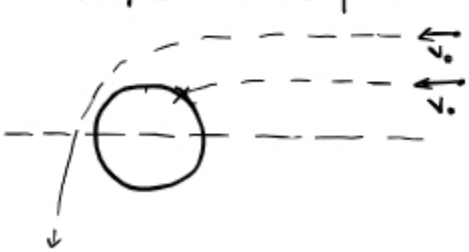


If the particles are moving towards the planet, obviously any approaching within the planet's maximum cross-sectional area πR^2 will hit the planet

Outside that, hitting the planet depends on if the gravitational force is sufficient to deflect the particle enough to hit the surface

Each dust particle has angular momentum $l = mv_0 b = mr^2\dot{\phi}$
where b is the distance from the particle's trajectory infinitely far from the planet (⇒ before deflection) and the planet's center

If when the particle scatters it comes within distance R of the planet center, it hits the planet



For each particle, the initial energy at ∞ is $E_i = \frac{1}{2}mv_0^2$

it barely grazes the surface if the final energy is exactly $E_f = \frac{1}{2}mv_f^2 - \frac{GMm}{R}$

gravitational potential energy at the surface

Since both energy and angular momentum are conserved:

grazing surface → $l_i = mv_0 b$ & $l_f = mv_f R$ ⇒ $v_0 b = v_f R$ ⇒ $v_f = v_0 \frac{b}{R}$

$$E_i = E_f \Rightarrow \quad \frac{1}{2}mv_0^2 = \frac{1}{2}mv_f^2 - \frac{GMm}{R}$$

$$\frac{1}{2}mv_0^2 = \frac{1}{2}m\left(v_0 \frac{b}{R}\right)^2 - \frac{GMm}{R}$$

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_0^2 \frac{b^2}{R^2} - \frac{GMm}{R}$$

$$\Rightarrow \frac{1}{2}mv_0^2 \frac{b^2}{R^2} = \frac{1}{2}mv_0^2 - \frac{GMm}{R}$$

$$\Rightarrow b^2 = \left(\frac{1}{2}mv_0^2 - \frac{GMm}{R}\right)\left(\frac{2R^2}{mv_0^2}\right)$$

$$b^2 = R^2 - \frac{2GMR}{v_0^2}$$

So any particle within distance b from origin will hit the planet

the area formed by this requirement is $\sigma = \pi b^2 = \pi R^2 - \frac{2\pi GMR}{v_0^2}$

Within this area all the dust particles are captured