J05M2 - Planet Moving Through Dust Cloud (Solution by Jim Wu)
A planet of mass M and radius R moves through a cloud of interplanetary dust at a constant velocity v_0. The dust particles have negligible mass. Depending on its initial position when the planet is still far away, each dust particle will either hit or miss the planet as it moves by. When they strike the planet, they stick. The capture cross section σ is defined as the transverse area within which all dust particles are captured. Compute σ. *Hint:* It is useful to consider the capture process in the reference frame of the planet.

Solution:

In the reference frame of the planet, which is assumed to very massive, a cloud of interplanetary dust is coming towards the planet from infinity with initial velocity v_0. Let's consider a small chunk of dust of mass Δm with impact parameter b.

Initially, the dust clump has an angular momentum of $L_i = \Delta m b v_0$. Following the trajectory, any chunk of dust with an orbit where the point of closest approach to the planet is, in fact, less than R will clearly collide with the planet. For the dust particles that just barely graze the surface of the planet, their final angular momentum is $L_f = \Delta m R v_f$, where v_f is the velocity just before sticking to the planet. By conservation of angular momentum,

$$bv_0 = Rv_f$$

Additionally, energy is conserved during the orbit:

$$\frac{1}{2}\Delta m v_0^2 = \frac{1}{2}\Delta m v_f^2 - \frac{GM\Delta m}{R}$$
$$\frac{1}{2}v_0^2 = \frac{1}{2}\left(\frac{bv_0}{R}\right)^2 - \frac{GM}{R}$$

Solving for the impact parameter, we get

$$b^2 = R^2 + \frac{2GMR}{v_0^2}$$

and the capture cross section is

$$\sigma = \pi b^2 = \pi\left(R^2 + \frac{2GMR}{v_0^2}\right)$$

Let's check some limits! A larger planet corresponds to more dust particles colliding into the planet, and so σ increases with R. A more massive planet means a stronger gravitational pull on the dust, which suggests that σ goes up with mass of the planet, M. Lastly, in the planet frame, if the dust particles move faster, they will have too much initial angular momentum and their orbit trajectory will not collide into the planet. Thus, σ should decrease with speed v_0.

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