Seth Dorfman          Prelims          January 7, 2006

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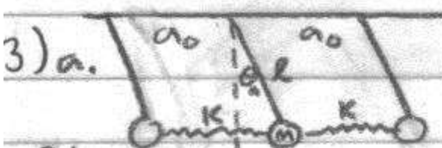
3) a.




$$L = \sum_n \tfrac{1}{2} m l^2 \dot\theta_n^2 - \tfrac{1}{2} K l^2 (\sin\theta_{n+1} - \sin\theta_n)^2 + m g l \cos\theta_n$$

$$\frac{\partial L}{\partial \theta_n} = K l^2 \left[-(\sin\theta_n - \sin\theta_{n-1})\cos\theta_n + (\sin\theta_{n+1} - \sin\theta_n)\cos\theta_n\right] - m g l \sin\theta_n$$

$$= K l^2 \cos\theta_n \left[\sin\theta_{n+1} - 2\sin\theta_n + \sin\theta_{n-1}\right] - m g l \sin\theta_n$$

$$\frac{\partial L}{\partial \dot\theta_n} = m l^2 \dot\theta_n \qquad \frac{d}{dt}\left[\frac{\partial L}{\partial \dot\theta_n}\right] = m l^2 \ddot\theta_n$$

$$m l \ddot\theta_n = K l \cos\theta_n \left[\sin\theta_{n+1} - 2\sin\theta_n + \sin\theta_{n-1}\right] - m g \sin\theta_n$$

For small oscillations $\psi_n = l\theta_n$, $\cos\theta_n \approx 1$ $\sin\theta_n \approx \theta_n$

$$m \ddot\psi_n = K\left[\psi_{n+1} - 2\psi_n + \psi_{n-1}\right] - \frac{mg}{l}\psi_n$$

b. Let w be the frequency of the modes $\theta_n \sim e^{iwt}$

$$-m w^2 \psi_n = K\left[\psi_{n+1} - 2\psi_n + \psi_{n-1}\right] - \frac{mg}{l}\psi_n$$

$$\psi_n\left[2Kl + mg - m l w^2\right] = K l\left[\psi_{n+1} + \psi_{n-1}\right]$$

$$\psi_n\left[2 + \frac{mg}{Kl} - \frac{m}{K} w^2\right] = \psi_{n+1} + \psi_{n-1}$$

$$\frac{1}{a_0}\psi_n\left[\frac{m}{K}\left(2\frac{K}{m} + \frac{g}{l} - w^2\right)\right] = \frac{\psi_{n+1} - \psi_n}{a_0} + \frac{\psi_n - \psi_{n-1}}{a_0}$$

Let $w_1^2 = \frac{g}{l}$, $w_2^2 = 2\frac{K}{m} + \frac{g}{l}$

$$\frac{1}{a_0}\psi_n\left[2\,\frac{w_2^2 - w^2}{w_2^2 - w_1^2}\right] = 2\frac{\partial \psi_n}{\partial x} \qquad \psi(x,t) \sim e^{i(Kx - wt)}$$

$$\frac{1}{a_0}\psi_n (w_2^2 - w^2) = iK\psi_n (w_2^2 - w_1^2)$$

$$w_2^2 - w^2 = iKa_0 (w_2^2 - w_1^2)$$

$$w^2 = w_2^2 - iKa_0 (w_2^2 - w_1^2)$$

$$w(K) = \sqrt{w_2^2 - iKa_0 (w_2^2 - w_1^2)}$$

c. From the dispersion relation, growth/damping is at a minimum when K is small (long wavelengts) and $w \sim w_2$. More specifically:

$$Ka_0 = -i\,\frac{w_2^2 - w^2}{w_2^2 - w_1^2}$$

| Frequency | $Ka_0$ | $\lvert Ka_0\rvert$ | result |
|---|---|---|---|
| $w > w_2$ | $+i\#$ | | damping |
| $w_1 < w < w_2$ | $-i\#$ | $< 1$ | growth |
| $w < w_1$ | $-i\#$ | $> 1$ | wavelength too small to propagate |

Thus waves can propagate only in the range:

$$\sqrt{\frac{g}{l}} \le w \le \sqrt{\frac{g}{l} + 2\frac{K}{m}}$$