May 2005 Preliminary Exam, Mechanics Problem 1

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Problem (Rope Around a Cylinder):

A long rope is wound around a cylinder of radius r so that the length l of the rope is in contact with the cylinder. The coefficient of static friction between the rope and the cylinder is μ_s. A force F is exerted on one end of the rope. For a given F, r, l, and μ_s, what force f must be applied to avoid the rope slipping? Explain why a small child can hold a large ocean liner in place using a device like this.

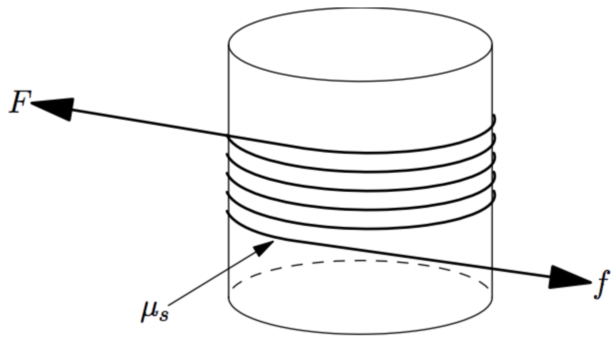

Figure 1: *Figure from J05M.1*

Reason for Solution:

Just another way to do the same problem.

Solution:

Map the problem to the x-axis, where force F acts in the negative direction at $x = 0$, and force f acts in the positive direction at $x = l$. We can derive a differential equation and solve using these boundary conditions. For a small mass δm, we can write down the equilibrium condition as follows:

$$\delta m \ddot{x} = 0 = T[x + dx] - T[x] + \mu_s N \tag{1}$$

To find the normal force, consider the component of the tension in the rope acting radially given some small angle around the length of the rope $d\theta = \frac{dx}{r}$:

$$\begin{aligned} N &= \sin\left(\frac{\theta}{2}\right)(T[x + dx] + T[x]) \\ &= \sin\left(\frac{dx}{2r}\right)(T[x + dx] + T[x]) \\ &\approx \frac{dx}{2r}(T[x + dx] + T[x]) \\ &\approx \frac{dx}{r}T[x] \end{aligned} \tag{2}$$

Therefore,

$$T[x+dx]-T[x]=-\mu_s N=-\frac{\mu_s dx}{r}T[x] \tag{3}$$

$$\implies \frac{T[x+dx]-T[x]}{dx}=-\frac{\mu_s}{r}T[x] \tag{4}$$

$$\implies \frac{dT[x]}{x}=-\frac{\mu_s}{r}T[x] \tag{5}$$

$$\implies T[x]=T[0]\exp\left(\frac{-\mu_s x}{r}\right) \tag{6}$$

Using the boundary condition at $x=0$ yields the following:

$$T[x]=F\exp\left(\frac{-\mu_s x}{r}\right) \tag{7}$$

At $x=l$, we can calculate the force f required for static equilibrium:

$$T[l]=f=F\exp\left(\frac{-\mu_s l}{r}\right)\ll F \tag{8}$$