1 January 2005, Electromagnetism, Problem 1

1.1 (a)

Outside the field is 0, by a known theorem of conductors that says that the field outside a spherical conductor is given simply by the total sum of the charges inside, which in this case is 0. Similarly, the field inside the inner sphere is 0 by another theorem of conductors that says that the field within an empty conductor with no charges inside it is 0. In between the spheres, we use the equations:

$$\nabla \cdot \boldsymbol{D} = \rho_f$$
$$\nabla \times \boldsymbol{D} = \epsilon \nabla \times \boldsymbol{E} = \boldsymbol{0}$$

There are no free charges anywhere in between the spheres, so we can write the displacement field as the gradient of a potential, and with:

$$\nabla^2 \Phi = 0 \quad \boldsymbol{D} = -\nabla \Phi$$

Let z be the direction that is perpendicular to the interface between the different dielectric regions (upward, in the picture). We can then solve the equation for Φ in spherical coordinates, with azimuthal symmetry. The most general solution turns out to be:

$$\Phi = \left(Ar^2 + \frac{B}{r}\right) cos\theta + \frac{C}{r}$$
$$\boldsymbol{D} = \left[\frac{C}{r^2} - \left(2Ar - \frac{B}{r^2}\right) cos\theta\right] \hat{r} + \left(Ar + \frac{B}{r^2}\right) sin\theta\hat{\theta}$$

Now we use the boundary conditions:

1) $Q_{out} = Q$

$$\sigma_{f_{out}}(\theta) = -D_r(b) = -\left[\frac{C}{b^2} - \left(2Ab - \frac{B}{b^2}\right) cos\theta\right]$$
$$Q_{f_{out}} = \int_0^{2\pi} \int_0^{\pi} \sigma_{f_{out}}(\theta) b^2 sin\theta \, d\theta \, d\phi = -4\pi C = -Q$$
$$C = \frac{Q}{4\pi}$$

2) $\Delta D_\theta(b) = 0$

$$-\left(Ab + \frac{B}{b^2}\right) sin\theta = 0$$
$$A = -\frac{B}{b^2}$$

3) $DeltaD_\theta(a) = 0$

$$A = -\frac{B}{a^2}$$
$$A = B = 0$$
$$\boldsymbol{D} = \frac{Q}{4\pi r^2}\hat{r} \tag{1}$$
$$\boldsymbol{E} = \frac{Q}{4\pi\epsilon r^2}\hat{r} \quad \theta < \pi/2 \tag{2}$$
$$\boldsymbol{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} \quad \theta > \pi/2 \tag{3}$$

1.2 (b)

We want the bound charges on the surfaces of the dielectric:

$$\boldsymbol{D} - \epsilon_0\boldsymbol{E} = \boldsymbol{P}$$
$$\boldsymbol{P}_{top} = \frac{Q}{4\pi r^2}(1 - \frac{\epsilon_0}{\epsilon})\hat{r}$$
$$\sigma_{b_{out}} = \boldsymbol{P_{top}} \cdot \hat{r} = \frac{Q}{4\pi b^2}\left(1 - \frac{\epsilon_0}{\epsilon}\right) \tag{4}$$
$$\sigma_{b_{in}} = -\boldsymbol{P_{top}} \cdot \hat{r} = -\frac{Q}{4\pi a^2}\left(1 - \frac{\epsilon_0}{\epsilon}\right) \tag{5}$$
$$\sigma_{b_{hor}} = -\boldsymbol{P_{top}} \cdot \hat{z} = 0 \tag{6}$$